MATH 9: ALGEBRA: FALLACIES, INDUCTION

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1. Quantifier Recap

First, a review of quantifiers introduced last week. Recall that $\exists$ is called the **existential quantifier** $\forall$ is called the **universal quantifier**. We write the statement $\forall x(P(x))$ to mean *for all values of x, $P(x)$ is true*, and $\exists x(P(x))$ to mean *there is some value of x such that $P(x)$ is true.* Here $P(x)$ is a predicate, as defined last week.

Generalized De Morgan's Laws:

$$\neg\forall x(P(x)) \leftrightarrow \exists x(\neg P(x))$$

$$\neg\exists x(P(x)) \leftrightarrow \forall x(\neg P(x))$$

Note that multiple quantifiers can be used in a statement. If we have a multivariable predicate like $P(x, y)$, it is possible to write $\exists x\exists y(P(x, y))$, which is identical to a nested statement, $\exists x(\exists y(P(x, y)))$. If it helps to understand this, you can realize that the statement $\exists y(P(x, y))$ is a logical predicate in variable x. Note that, in general, the order of the quantifiers **does** matter.

To negate a statement with multiple predicates, you can do as follows:

$$\neg\forall x(\exists y(P(x, y)))$$

$$= \exists x(\neg\exists y(P(x, y)))$$

$$= \exists x(\forall y(\neg P(x, y))).$$

2. Negations

$$\neg(A \implies B) \leftrightarrow A \wedge \neg B$$

$$\neg(A \leftrightarrow B) = (\neg A \leftrightarrow B) = (A \leftrightarrow \neg B) = (A \oplus B)$$

Here I am using $=$ to indicate logical equivalence, just because it is a little less ambiguous when the statements themselves contain the $\leftrightarrow$ sign; ultimately, the meaning is the same. The $\oplus$ symbol is the xor logical relation, which means that exactly one of A, B is true, but not both.

3. Proofs

Conditional proof: To prove $A \implies B$, assume A and then deduce B.

Proof by contradiction: To prove some logical statement or theorem T, assume that T is false (or assume that the negation of T is true) and deduce a logical contradiction (e.g. a statement of the form $A \wedge \neg A$).

To prove a statement of the form $\exists x(P(x))$, it is enough to find one example of x for which $P(x)$ holds true.

To prove a statement of the form $\forall x(P(x))$, you must provide a general argument which is logically valid for all possible values of x, that deduces that $P(x)$ is true.

To prove any statement with a quantifier, it may sometimes be easier to use a proof by contradiction. For example, given $0 \neq 1$, can you prove $\forall x(x \neq 0 \vee x \neq 1)$? Try writing out the logical negation of the statement and go from there.

4. Fallacies

A fallacy is reasoning that is evaluated as logically incorrect. Fallacy vitiates the logical validity of the argument and warrants its recognition as unsound.

Fallacies are common, and found everyday in the world around us - news headlines and articles, regular conversation, even sometimes in books or things we are taught, and (of course) - quite commonly - in online fora (esp social media comment threads, but I'll leave it to you to explore on your own). Wikipedia also has an excellent list of logical fallacies article.

Each of the logical fallacies presented here can be written in purely logical form, using logical language. The result will be a false statement, for example $A \vdash \neg A$. I will present each logical fallacy in name and then provide examples of the fallacious argument - see if you can figure out the logical form of the fallacy. In the next section I will write out the answers.

In the context of logical argumentation, a deductive statement of the form $A \vdash B$ is also a false statement if A and B are variables for any logical statement, because one statement cannot in general be used to deduce an unrelated one.

Formal fallacies. A formal fallacy is essentially an invocation of a deductive syllogism that does not exist; it is a type of non-sequitur argument.

Argument from Fallacy.

- She says he's annoying but she never thinks logically, therefore he's not annoying.
- If you don't even try to aim your arrow, of course you're not going to hit the target.
- I built a cute weather machine made out of ramen and it's giving me a reading that it'll rain today - are you purposefully not bringing your umbrella just to make fun of me?

Conjunction fallacy. This one has had formal psychological studies conducted that show it is quite common thinking.

- I think it's more likely that she's a political activist and lawyer than that she's a lawyer.
- I refuse to believe you will make it on time, but I'm willing to believe you'll make it on time if you actually sleep on time for once.
- You can't just have a bunch of different words for the same fruit. But maybe a country that grows a lot of them might.

Masked-man fallacy. Tbh I don't know how common this one is but it's a useful exercise to understand it (i.e. whether or not it's a common fallacy, you should have the logical skills to dissect it).

- According to the footage, the criminal had a white jacket on. I've seen your wardrobe and you don't have a white jacket, so it couldn't have been you.
- I can recognize Perry the Platypus, but that platypus is not wearing a hat, it is therefore not Perry the Platypus.
- I know orange doesn't know how to do electrical tasks, but someone did do them, so it couldn't have been orange that did them.

Inductive fallacy.

- I have never had an issue with corporations polluting water in my town, therefore I know it's not an issue in other towns in my country.
- I saw a beetle come back to life once, so I know that insects have resurrection powers!
- Everyone I know who wears glasses is smart, and I see you're wearing glasses so you must be smart too.

Syllogistic fallacies. Syllogistic fallacies are formal fallacies whose form imitates, or very closely resembles, a valid syllogism. This can lead to confusion if the reader or interpreter is unable to see, or pick apart, the precise logical nature of the argument. I won't list all possible examples here (it is not educational or illustrative to do so), but instead say this: given any logical argument, you should be able to interpret or write the argument using formal logical language, then you'll be able to see what's going on. Here's one example of a syllogistic fallacy, see if you can break it down: "Everyone who goes to the gym regularly is fit. None of my friends go to the gym regularly, so none of them are fit."

Propositional fallacies. These come from misunderstanding a logical connective. Logical connectives, like $\implies$ and $\vee$, have strict meanings, and it is fallacious to make conclusions that don't respect their definitions.

Affirming a disjunct.

- The Prime Minister's interpreters should be able to translate from Korean; I know one of them learned Korean, so the other ones probably don't know Korean.
- Every president of the club is popular or intelligent - I know she's popular, so I doubt she's intelligent.
- I know that soap cleans things but this soap worked in the dishwasher so it's not gonna work on your clothes.

Affirming the consequent.

- Coming from an intelligent family really increases your chances of success. You did very well on the exam, it must be in your genes.
- A good leader can greatly reduce a country's unemployment. Unemployment rates went down in that country, so their leader should be thanked.
- I paid my nephew to do the dishes and I see the sink's clean now, he did a good job.

Denying the antecedent. Jsyk, 'antecedent' and 'consequent' are related words, they refer to the left and right propositions in an implication statement: *antecedent* $\implies$ *consequent*.

- If a liquid that is calm at room temperature is bubbling, that means it is extremely hot. The oil is not bubbling, therefore it is not hot.
- Low barometric pressure typically indicates an incoming storm. The pressure today is at normal levels, so no storm is coming.
- They're always nice when they're happy. But they're not happy today so you may want to avoid them.

Quantification fallacy. A trivially-applied quantifier is used for an inappropriate deduction or implication.

Existential fallacy.

- The light from stars in the solar system other than our sun is harmful for phytoplankton.
- Each and every student currently in my classroom passed the exam.
- I'm your favorite math 9 teacher.

Probabilistic fallacies. When statistical understandings are misinterpreted. These fallacies involve violating facts or theorems related to statistics. Poor statistical intuition is actually extremely common: look up *base rate fallacy* and *law of truly large numbers* (also called *the likelihodd of unlikely events*).
Informational fallacies. Fallacious judgments of information. Sometimes poor or incorrect information is used to make a conclusion; sometimes a person underestimates the amount of information required to make a conclusion; sometimes a person does not understand what types of information are required to make a conclusion (for example, look up *jumping to conclusions*); etc.

As far as logical and cognitive literacy goes, there is a lot more to study, especially less mathematically related topics, like cognitive biases (Wikipedia also has a great list on that), you can ask me if you are interested but we won't cover them in class.

5. Fallacies: Answers

Here is the logical form of each fallacy, with some brief explanation.

Argument from Fallacy. $(A \implies B) \wedge (\neg A) \vdash \neg B$. You may laugh at me and my ramen machine, but it might still rain, and then I promise you I will laugh at you. In general, if an argument for a conclusion is false, the conclusion might still be true.

Conjunction fallacy. $A \vdash A \wedge B$. This happens when people think than an 'ordinary' occurrence of an event is less likely than a 'special' occurrence - for example, maybe someone is so politically active that nobody ever thought she would become a lawyer, and that's why they say they're willing to believe that she's an activist and lawyer than just a lawyer. But if she's an activist and lawyer then she is still a lawyer, so 'she is a lawyer' is true. But the statement 'she is a lawyer' pays no mention to the most salient trait of her personality, which is possibly why people are less inclined to believe it.

Masked-man fallacy. $(A \vdash X) \vdash X$. This fallacy happens when someone draws an objective conclusion from a particular person's thinking. If I believe you are not the criminal, this does not mean for certain that you are not the criminal. The correct, non-fallacious deduction is "I know that you are not the criminal" - this statement is true, but because my knowledge has an error in it, the objective conclusion is false.

Inductive fallacy. $(\exists x(P(x))) \vdash (\forall x(P(x)))$. When a phenomenon is observed in some cases but a conclusion is drawn for all cases, this is a fallacy. My town is not representative of all towns in the country; my experience is not representative of every person's experience; etc. It is common to find this experience when one person draws an inference from one context to a conclusion from a starkly different context, e.g. a man believes that because he never notices sexism, that women never experience it.

Syllogistic fallacies. For the particular argument given, the logical form is $((A \implies B) \wedge (C \implies \neg A)) \vdash (C \implies \neg B)$. This is clearer to understand if you rewrite it slightly, let's use $D = \neg C$ and take the contrapositive of the second and third implications, to get $((A \implies B) \wedge (A \implies D)) \vdash (B \implies D)$.

Affirming a disjunct. $((A \vee B) \wedge A) \vdash \neg B$. This fallacy often occurs if you think that a result that had multiple possible causes is known to have occurred from one cause, that all other causes are false. In reality, multiple of the causes might be true in some way or another. Maybe all of the PM's interpreters know Korean. In general, the thing to keep in mind here is simply that if a statement $A \vee B$ is true, it's entirely possible that both A and B are true.

Affirming the consequent. $((A \implies B) \wedge B) \vdash A$, or $(A \implies B) \vdash (B \implies A)$. If an implication is true and the result is true, that does not mean that the original supposed 'cause' is also true. A result can be true for other causes. Maybe I did well on the exam because I study well and I pay attention in class, and in actuality I inherited the genes of a photosynthetically challenged phytoplankton. On the other hand, maybe I study well and pay attention in class and certain other factors such as educational privilege also contribute to my success. One cause cannot rule out others; if an implication is true, it does not mean the result is equivalent to the cause. (And, it turns out, my nephew didn't do any job at all, he simply paid his lil bro to do the dishes. Just because I invested money in the implication does not make the fallacy more logical.)

Denying the antecedent. $((A \implies B) \wedge \neg A) \vdash \neg B)$ or $(A \implies B) \vdash (\neg A \implies \neg B)$. I read a story once of some college students who wanted to deep fry some vegetables so they put a pot of oil on the stove to heat up. They thought that because the oil looked totally calm and motionless, that it wasn't hot yet. Fortunately they were caught by a flatmate before they set the house on fire. As for the third example, people who act nice when they're happy can still act nice when they're sad (though this, like much of logic, is a basic skill that eludes a lot of people).

Existential fallacy. $\forall x(P(x)) \vdash \exists x(P(x))$. I'm your only math 9 teacher, just because you like me more than all your other math 9 teachers does not mean you like me more than anyone. And if the light from stars other than our sun is harmful for phytoplankton, that's a practically meaningless statement, because there are no other stars in our solar system, so our phytoplankton are safe. And that teacher who said that every student in their classroom passed the exam - it was veteran's day, no students were in the school at the time. Three kids failed.

6. Mathematical Induction

The **Principle of Mathematical Induction** is an extremely famous and useful mathematical argument, first formalized by mathematician Peano, in what was also the first formalization of modern logic. It's the idea that if you start at 0 and keep hopping upwards by $+1$, that eventually you will reach any natural number. Formally, it is as follows:

Let $P(n)$ be a predicate on natural numbers (nonnegative integers). Suppose that there is a mathematical proof that $P(0)$ is true, and there is also a mathematical proof that $P(n) \implies P(n+1)$ is true. Then $P(n)$ is true for all n.

Proving that $P(0)$ is true is called the **base case**. You can use numbers other than 0 for the base case, as long as you have some base case then the final result will hold for all integers greater than or equal to the base number.

Proving the implication $P(n) \implies P(n+1)$ is called the **inductive step**. It is important to understand that it is the implication itself you are proving, not either of the statements $P(n)$ or $P(n+1)$. In order to prove the implication, you assume that $P(n)$ is true and then prove $P(n+1)$ based on that assumption. The assumption of $P(n)$ is called the **inductive hypothesis**. (It is sometimes useful to assume that $P(n)$ holds for all values between the base case and n, which is called the **strong inductive hypothesis**.)

If you think about it, you should be able to get an intuition for why the principle of mathematical induction is true. Formally there are two ways to explain it:

- You can prove the principle of mathematical induction from the principle of well-ordered natural numbers. This principle states that every nonempty subset of natural numbers has a minimum element. To see why this proves the induction principle, prove by contradiction: suppose $P(0)$ and $P(n) \implies P(n+1)$ is true, but $\exists r(\neg P(r))$. Then $S = \{n | \neg P(n)\}$ is nonempty; let the minimum of S be s, then $P(s)$ is false but $P(s-1)$ is true, which is a contradiction.
- You can simply define the natural numbers as the set of all numbers for which the inductive principle holds true. This is the way that Peano did it. In this model, depending on how you configure the settings, it is possible to have 'nonstandard' natural numbers (in which the well-ordered principle is false), but we won't go into that in this class.

7. Laws, Equivalences

A recap of logical equivalences.

Commutative laws:

1. $A \wedge B = B \wedge A$
2. $A \vee B = B \vee A$
3. $A \leftrightarrow B = B \leftrightarrow A$

Associative laws:

1. $A \wedge (B \wedge C) = (A \wedge B) \wedge C$
2. $A \vee (B \vee C) = (A \vee B) \vee C$
3. $A \leftrightarrow (B \leftrightarrow C) = (A \leftrightarrow B) \leftrightarrow C$

Distributive laws:

1. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
2. $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
3. $A \implies (B \wedge C) = (A \implies B) \wedge (A \implies C)$
4. $A \implies (B \vee C) = (A \implies B) \vee (A \implies C)$

Codistributive laws:

1. $(A \wedge B) \implies C = (A \implies C) \vee (B \implies C)$
2. $(A \vee B) \implies C = (A \implies C) \wedge (B \implies C)$
3. $\neg(A \wedge B) = (\neg A) \vee (\neg B)$, de Morgan's law
4. $\neg(A \vee B) = (\neg A) \wedge (\neg B)$, de Morgan's law
5. $\neg\exists x(P(x)) = \forall x(\neg P(x))$
6. $\neg\forall x(P(x)) = \exists x(\neg P(x))$
7. $\neg(\neg A) = A$

Implication laws:

1. $A \implies B = B \vee (\neg A)$
2. $\neg(A \implies B) = A \wedge (\neg B)$
3. $A \implies B = (\neg B) \implies (\neg A)$, contrapositive
4. $A \leftrightarrow B = (A \implies B) \wedge (B \implies A)$
5. $A \leftrightarrow B = (\neg A) \leftrightarrow (\neg B)$
6. $\neg(A \leftrightarrow B) = A \oplus B$

Transitivity laws (these are not equivalences, but useful for deduction):

1. $(A \implies B) \wedge (B \implies C) \vdash (A \implies C)$
2. $(A \leftrightarrow B) \wedge (B \leftrightarrow C) \vdash (A \leftrightarrow C)$
3. $(A \oplus B) \wedge (B \oplus C) \vdash (A \leftrightarrow C)$

8. Properties of Rational Numbers

The following statements are true for all rational numbers. The $\forall a \in \mathbb{Q}$ quantifiers will be omitted for whatever variables appear in the statements. (Existential quantifiers, where needed, will be left in.)

Ordering.

1. Order properties: $\neg(a < a)$, $(a < b) \implies \neg(b < a)$, and $(a < b) \wedge (b < c) \implies (a < c)$.
2. Total order: $a = b \vee a < b \vee b < a$.
3. Dense order: $a < b \implies (\exists c \in \mathbb{Q}(a < c < b))$.
4. Archimedean property: $a, b \neq 0 \implies \exists n \in \mathbb{Z}(nb > a)$. Given any rational number target and any other nonzero rational number step size, it is possible to reach the target from an integer numbero f steps.

Addition and subtraction.

1. Commutative: $a + b = b + a$
2. Associative: $a + (b + c) = (a + b) + c$
3. Identity: $a + 0 = 0 + a = a$
4. Inverse: given a, $\exists b(a + b = 0)$. This b is called the additive inverse of a, and is usually written $-a$.
5. Subtraction: subtraction $a - b$ is defined as $a + (-b)$.
6. Order cancellation: $a < b \leftrightarrow a + c < b + c$.

Multiplication and division. Multiplication and division are written with various symbols, but I will write multiplication between variables without a symbol (e.g. ab is a times b), multiplication with numbers using a dot (e.g. $2 \cdot 2 = 4$), and division as fractions.

1. Commutative: $ab = ba$
2. Associative: $a(bc) = (ab)c$
3. Distributivity: $a(b + c) = ab + ac$
4. Identity: $a \cdot 1 = 1 \cdot a = a$
5. Inverse: given $a \neq 0$, $\exists b(ab = 1)$. This b is called the multiplicative inverse of a, and is usually written $\frac{1}{a}$ or a^{-1}.
6. Division: division $\frac{a}{b}$ is defined as $a \cdot \frac{1}{b}$, also written ab^{-1}.