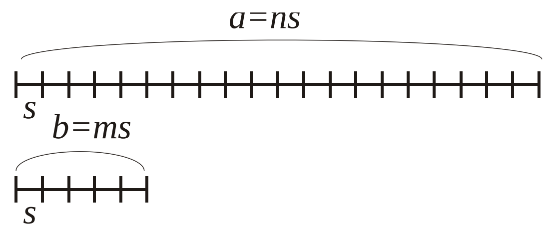
September 27, 2020

Geometry.

Commensurate and incommensurate segments. The Euclidean algorithm.

Definition. Two segments, a and b, are **commensurate** if there exists a third segment, s, such that it is contained in each of the first two segments a whole numbers of times with no remainder.



The segment s is called a common measure of the segments a and b. The concept of commensurability is similar to that of the common divisor for integers. It can be extended to any two quantities of the same denomination – two angles, two arcs of the same radius, or two weights.

The greatest common measure.

If a common measure s of two segments a and b is sub-divided into two, three, or, generally, any number of equal smaller segments, these smaller segments are also common measures of the segments a and b. In this way, an infinite set of common measures, decreasing in length, can be obtained, – }. Since any common measure is less than the smaller segment, b, there must be the largest among the common measures, which is called the greatest common measure.

Finding the greatest common measure (**GCM**) is done by the method of consecutive exhaustion called **Euclidean algorithm.** It is similar to the method of consecutive division used for finding the greatest common divisor in arithmetic. The method is based on the following theorem.

Theorem. Two segments a and b are commensurate, if and only if the smaller segment, b, is contained in the greater one a whole number of times with no remainder, or with a remainder, $r < b$, which is commensurate with the smaller segment, b.

$$\exists n \in N: (a = nb + r) \wedge \left((r = 0) \vee \left(\exists s, \{p, q\} \in N: (b = ps) \wedge (r = qs)\right)\right).$$

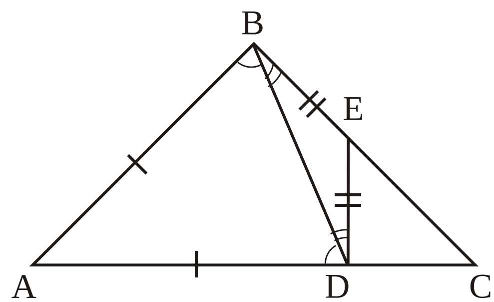
The greatest common measure of two segments is also the greatest common measure of the smaller segment and the remainder, or there is no remainder.

Proof. First, consider the necessary condition. Let a and b be commensurate, $\{\exists s, \{n, m\} \in N: (a = ns) \wedge (b = ms)\}$, and $a > b$. Let s be their greatest common measure. Then, either $s = b$ $(m = 1)$ and segment b is contained in a a whole number of times with no remainder, being the GCM of the two segments, or, $\exists k \in N: a = kb + r, 0 < r < b$. Then, $a = ns = kb + (n - km)s$, where $m < km < n$, and, therefore, $r = qs, \{q = n - km\} \in N$, which shows that r and b are commensurate. The sufficiency follows from the observations that (i) if segment b is contained in a a whole number of times with no remainder, then the segments are commensurate, and b is the greatest common measure of the two, while (ii) if $a = kb + r$, and b and r are commensurate with the greatest common measure s, $\exists\{p, q\} \in N: (b = ps) \wedge (r = qs)$, then $a = (kp + q)s = ns, n = kp + q \in N$, and a and b are also commensurate with the same GCM.

The Euclidean algorithm.

In order to find the GCM of the two segments, a and b, we can proceed as follows. First, using a compass exhaust the greater segment, marking on it the smaller segment as many times as possible, until the remainder is smaller than the smaller segment, b, or there is no remainder. According to Archimedes' exhaustion axiom, these are the only two possible outcomes. Following the above theorem, the problem now reduces to finding the GCM of this remainder, r_1, and the smaller segment, b. We now repeat the same procedure, exhausting segment b with r_1, and again, there is either no remainder and r_1 is the GCM of a and b, or there is a remainder $r_2 < r_1$. The problem is then reduced to finding the GCM of a pair of even smaller segments, r_1 and r_2, and so on. If segments a and b are commensurate and their GCM, s, exists, then this process will end after some number of steps, namely, on step n where $r_n = s$. Indeed, all remainders in this process are multiples of s, $\forall m: r_m = p_m s, p_m \in N$ and $p_1 > p_2 > \cdots > p_m > \cdots$ is the decreasing sequence of natural numbers, which necessarily terminates, since any non-empty set of positive integers has the smallest number ("principle of the smallest integer"). If the procedure never terminates, then segments a and b have no common measure and are incommensurate.

Example. The hypotenuse of an isosceles right triangle is incommensurate to its leg. Or, equivalently, the diagonal of a square is incommensurate to its side.



Proof. Consider the isosceles right triangle ABC shown in the Figure. Because the hypotenuse is less than twice the leg by the triangle inequality, the leg can only fit once in the hypotenuse, this is marked by the segment AD. Let the perpendicular to the hypotenuse at point D intercept leg BC at point E. Triangle BDE is also isosceles. This is because angles BDE and DBE supplement equal angles ADB and ABD to 90 degrees, and therefore are also equal. Triangle CDE is an isosceles right triangle, similar to ABC. Its leg |DC|=|AC|-|AB|=|DE|=|BE| is a remainder of subtracting the leg |AB|=|AD| from the hypotenuse, |AC|, while the hypotenuse, |CE| =|BC|-|BE|=|BC|-|DC|, is the remainder of subtracting this remainder from the leg |AB|=|BC|. Hence, on the second step of the Euclidean algorithm we arrive at the same problem as the initial one, only scaled down by some overall factor. Obviously, this process never ends, and therefore the hypotenuse |AC| and the leg |AB| are incommensurate.