September 20, 2020

Baseline review test. Algebra.

1. Open brackets and expand the following expressions
   a. $(a+b)^2 =$
   b. $(a-b)^2 =$
   c. $(a+b)^3 =$
   d. $(a-b)^3 =$
2. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^2 + b^2 =$
   c. $a^3 - b^3 =$
   d. $a^3 + b^3 =$
   e. $1 + a + a^2 + a^3 =$
3. For a quadratic equation $ax^2 + bx + c = 0$ the roots are,
   $x_{1,2} =$
   and they have the following properties,
   $x_1 + x_2 =$
   $x_1 \cdot x_2 =$
4. Open brackets and expand the following expression

   $(a+b)^{10} =$

5. What is the number of permutations of n objects?
6. How many ways is there to select k objects out of n if,
   a. order does matter?
   b. order does not matter?
7. Write the formula for a binomial coefficient

   $${}_nC_k \equiv C_n^k \equiv \binom{n}{k} =$$

   and explain its relation to combinatorics and certain counting problems.

Solutions to the baseline review test. Algebra.

1. Open brackets and expand the following expressions
   a. $(a+b)^2 = a^2 + 2ab + b^2$
   b. $(a-b)^2 = a^2 - 2ab + b^2$
   c. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
   d. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
2. Factor the following expressions
   a. $a^2 - b^2 = (a-b)(a+b)$
   b. $a^2 + b^2 = \ldots$
   c. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
   d. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
   e. $1 + a + a^2 + a^3 = (1+a)(1+a^2) = \frac{1-a^4}{1-a}$

   The last example was a particular case of a geometric progression, whose sum is one of the most important expressions in algebra,

$$1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$$

   A simple heuristic way to prove this result is to multiply both sides with $(1-a)$ and then open the brackets,

$$(1 + a + a^2 + \cdots + a^n)(1-a) = 1 - a + a - a^2 + a^2 - \cdots - a^n + a^n - a^{n+1} = 1 - a^{n+1}$$

3. For a quadratic equation $ax^2 + bx + c = 0$ the roots are,
   $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   and they have the following properties,
   $x_1 + x_2 = -\frac{b}{a}$
   $x_1 \cdot x_2 = \frac{c}{a}$

Although this can be checked by direct substitution of the formula for $x_{1,2}$, the easiest way to see this is by rewriting the equation in the reduced form and identifying it with the product of the two brackets,

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Leftrightarrow (x - x_1)(x - x_2) = 0 \Leftrightarrow x^2 - (x_1 + x_2)x + x_1x_2 = 0$$

This is an example of the polynomial factorization, which we will be studying in significant detail this year.

4. Open brackets and expand the following expression,

   $(a+b)^n = a^n + na^{n-1}b \ + {}_nC_k a^{n-2}b^2 + {}_nC_k a^{n-k}b^k + \cdots + {}_nC_1 a\, b^{n-1} + b^n$

This is the Newton's binomial formula, and we will be reviewing and re-deriving it later this year using the mathematical induction.

5. What is the number of permutations of n objects? Answer: $n!$

This is the number of ways that n different objects (or subjects) can be placed into n different places.

Examples:

- How many ways is there to sit n people in a movie theater with n numbered chairs?
- How many ways is there to hand out n different books to n students?
- How many ways is there to place n numbered billiard balls into n numbered spots?

There is n ways to select a place for the first object (subject), for each of these n choices there is $n-1$ choice to place the second one, so there are $n(n-1)$ in total different choices to fill the first two spots, and so on. Hence, there are $n! = n(n-1)(n-2) \ldots \cdot 2 \cdot 1$.

6. How many ways is there to select k objects out of n if,
   a. order does matter? Answer: ${}_nP_k = \frac{n!}{(n-k)!}$
   b. order does not matter? Answer: ${}_nC_k = \frac{n!}{k!(n-k)!}$

Math 8-9 placement test 2015

1. How many ways are there to choose a team captain and 6 team members out of 15 candidates?
2. If x_1, x_2 are roots of the square equation $x^2 + 2x - 7$, what is $x_1 x_2$? $\frac{1}{x_1} + \frac{1}{x_2}$?
3. Simplify the following expression

$$\frac{(a^2 - b^2)^3}{(a-b)(a+b)^2}$$

4. How many "words" can you form by permuting the letters of the word "letter"? (A "word" is any combination of letters, not necessarily meaningful)
5. Points $A = (0,0)$, $B = (2,0)$, and C on the coordinate plane form an equilateral triangle. What are the coordinates of point C?
6. Factor $a^4 - b^4$.
7. Corners of a square with the side a are cut off so that a regular octagon is obtained. Find the area of this octagon.
8. Solve the inequality

$$\frac{x+5}{x^2 - 2x - 3} > 0$$

9. If we write the polynomial $(x+2)^{10}$ in the usual form $x^{10} + a_1 x^9 + a_2 x^8 + \ldots$, what would be the coefficient of x^6?
10. Find all integer numbers which give remainder 2 upon division by 7 and remainder 5 upon division by 13.
11. Given triangle ABC, explain how to construct (using ruler and compass) a point which is at equal distance from points A, B, and C.

Math 8-9 placement test 2015: solutions to selected problems

(1) $15 \times C_{14}^{6} = 15 \cdot \frac{14!}{6! \cdot 8!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$

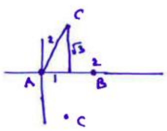
$= 15 \cdot 7 \cdot 13 \cdot 11 \cdot 3 = \boxed{45045}$

(2) $x_1 + x_2 = -2$, $\boxed{x_1 \cdot x_2 = -7}$ $\quad \frac{x_1 + x_2}{x_1 \cdot x_2} = \frac{1}{x_1} + \frac{1}{x_2} = \frac{-2}{-7} = \boxed{\frac{2}{7}}$

(3) $\frac{(a-b)^3 (a+b)^3}{(a-b)(a+b)^2} = \boxed{(a-b)^2 (a+b)}$

(4) letter $\quad \boxed{\frac{6!}{2! \cdot 2!}} = \frac{120 \cdot 6}{2 \cdot 2} = 60 \cdot 3 = \boxed{180}$

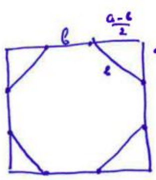
(5)



$\boxed{C = (1, \sqrt{3}) \text{ or } (1, -\sqrt{3})}$

(6) $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = \boxed{(a-b)(a+b)(a^2 + b^2)}$

(7)



$b^2 = 2\left(\frac{a-b}{2}\right)^2$

$2b^2 = (a-b)^2$

$b^2 + 2ab - a^2 = 0 \quad b = -a \pm \sqrt{a^2 + a^2}$

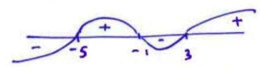
$b = (\sqrt{2} - 1)a$

$\text{Area} = a^2 - 4 \cdot \left(\frac{a-b}{2}\right)^2 \cdot \frac{1}{2} = a^2 - \frac{(a-b)^2}{2}$

$= a^2 - \frac{a^2}{2}(2 - \sqrt{2})^2 = a^2 - a^2[2 - 2\sqrt{2} + 1] = a^2[2\sqrt{2} - 2]$

$\boxed{\text{Area} = 2(\sqrt{2} - 1)a^2}$

(8) $\frac{x+5}{x^2 - 2x - 3} > 0 \quad \frac{x+5}{(x-3)(x+1)} > 0 \quad x \neq 3, \; x \neq -1$



$\boxed{x \in (-5, -1) \cup (3, +\infty)}$

(9) $(x+2)^{10} = x^{10} + \ldots + a_4 x^6 + \ldots \quad a_4 = C_{10}^{4} = \frac{10!}{6! \cdot 4!}$

$a_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 210 \quad \boxed{a_4 = 210}$

alternatively by Pascal's triangle

(10) $n = 7k + 2 = 13m + 5$

$7k = 3 \mod 13 \qquad 7k' = 1 \mod 13$

$k' = 2$

$k = 6 + 13\ell \quad (m = 7\ell + 3) \qquad 7 \cdot 6 = 3 \mod 13$

$n = 7(6 + 13\ell) + 2 = 7 \cdot 13\ell + 44$

$\boxed{n = 91\ell + 44 \quad \ell = 0, \pm 1, \pm 2 \ldots}$

12.

Math 9 placement test 2014

1. Open brackets and expand the following expressions
    a. $(a+b)^2 =$
    b. $(a-b)^3 =$
2. Factor the following expressions
    a. $a^2 - b^2 =$
    b. $a^3 - b^3 =$
    c. $a^3 + b^3 =$
    d. $1 + a + a^2 + a^3 =$
3. Find the coefficient of x^5 in the expression $(1+2x)^8$
4. Find the remainder of 3^{2014} upon division by 7.
5. Solve the equation

$$\frac{x^2+1}{2x} + \frac{2x}{x^2+1} = 2$$

6. Eight teams have reached the quarter-finals of the soccer World Cup.
    a. How many ways are there for these teams to be paired to play the quarter-final games?
    b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?
7. Corners of a square with the side a are cut off so that a regular octagon is obtained. Find the area of this octagon.

Math 9 placement test 2014: solutions to selected problems

1. Open brackets and expand the following expressions
   a. $(a+b)^2 =$
   b. $(a-b)^3 =$
2. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^3 - b^3 =$
   c. $a^3 + b^3 =$
   d. $1 + a + a^2 + a^3 =$
3. Find the coefficient of x^5 in the expression $(1+2x)^8$

$$(1+2x)^8 = \cdots + C_8^3(2x)^5 + \cdots = \cdots + \frac{8!}{5!\cdot 3!}\cdot 2^5 \cdot x^5 + \cdots = \cdots + 7\cdot 8\cdot 32\cdot x^5 + \cdots$$

4. Find the remainder of 3^{2014} upon division by 7.

$3^{2014} = (7+2)^{1007} = (\dots)\cdot 7 + 2^{1007} = (\dots)\cdot 7 + 4\cdot(7+1)^{335} = (\dots)\cdot 7 + 4$

5. Solve the equation

$$\frac{x^2+1}{2x} + \frac{2x}{x^2+1} = 2$$

$\frac{x^2+1}{2x} = t$, then $\frac{x^2+1}{2x} + \frac{2x}{x^2+1} = 2 \Leftrightarrow t + \frac{1}{t} = 2 \Leftrightarrow t^2 - 2t + 1 = 0 \Leftrightarrow t = 1 \Leftrightarrow \frac{x^2+1}{2x} = 1 \Leftrightarrow x^2 - 2x + 1 = 0 \Leftrightarrow x = 1.$

8. Eight teams have reached the quarter-finals of the soccer World Cup.
   a. How many ways are there for these teams to be paired to play the quarter-final games?

   For 2n teams, let the number of pairings be P_n. There is 2n-1 ways to pair the first team and thus select the first pair. For the remaining 2n-2 teams, there will be 2n-3 ways to select the second pair, and so on. Hence, $P_n = (2n-1)\cdot P_{n-1} = (2n-1)\cdot(2n-3)\cdot \dots \cdot 3\cdot 1$. For the case of 8 teams, $P_4 = 7\cdot 5\cdot 3 = 105$.

   b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?

   This is given by the number of possible ways to select 3 out of 8, where order matters. The answer is $A_8^3 = \frac{8!}{5!} = 6\cdot 7\cdot 8 = 336$.

9. Corners of a square with the side a are cut off so that a regular octagon is obtained. Find the area of this octagon.