April 11, 2021 Math 9

Geometry.

Trigonometry homework review.

The following trigonometric formulas will be useful for solving the homework.

1. Products of sine and cosine

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$$

$$\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta)-\sin(\alpha-\beta))$$

2. Sums of sine and cosine

$$\cos(\alpha)+\cos(\beta) = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos(\alpha)-\cos(\beta) = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin(\alpha)+\sin(\beta) = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin(\alpha)-\sin(\beta) = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

3. Sine and cosine of double and triple angle

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2\cot\alpha}{1+\cot^2\alpha}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{\cot^2\alpha-1}{\cot^2\alpha+1}$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

Solutions to selected homework problems:

1. Find the sum of the following series,

$$S = \cos x + \cos 2x + \cos 3x + \cos 4x + \cdots + \cos Nx$$

(hint: multiply the sum by $2\sin x/2$)

Solution 1: Easy way of summing the trigonometric series is by multiplying and dividing it with $\sin\frac{x}{2}$,

$$S\,\frac{\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}(\cos x+\cos 2x+\cdots+\cos nx)}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}\cos x+\sin\frac{x}{2}\cos 2x+\cdots+\sin\frac{x}{2}\cos nx}{\sin\frac{x}{2}} =$$
$$\frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\frac{3x}{2}-\sin\frac{3x}{2}+\sin\frac{5x}{2}-\sin\frac{5x}{2}+\cdots-\sin\left(n-\frac{1}{2}\right)x+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} = \frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} =$$
$$\frac{\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

Solution 2. A different and perhaps easier way of summing the above trigonometric series is by adding the expression for S_1, or S_2, rearranged from back to front, to itself , as we did when summing the arithmetic series,

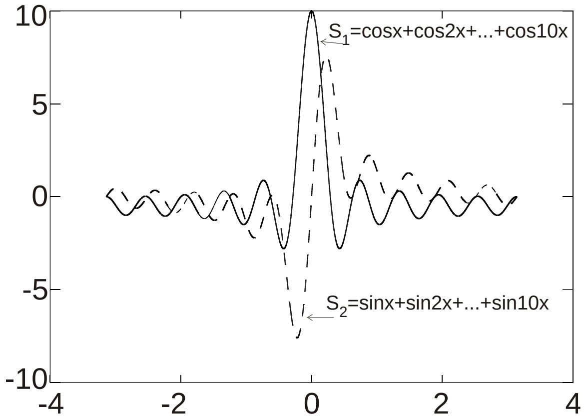
$$S_1 = \cos x + \cos 2x + \cdots + \cos nx$$

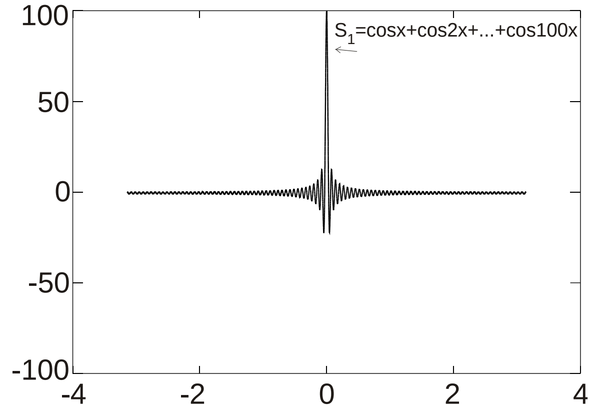
$$S_1 = \cos nx + \cos(n-1)x + \cdots + \cos x$$

Wherefrom,

$$\mathrm{S}_1 = \frac{1}{2}\big((\cos \mathrm{x} + \cos \mathrm{nx}) + (\cos 2\mathrm{x} + \cos(\mathrm{n}-1)\mathrm{x}) + \cdots + (\cos \mathrm{nx} + \cos \mathrm{x})\big) =$$
$$\cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\left(\cos(\mathrm{n}-1)\frac{\mathrm{x}}{2} + \cos(\mathrm{n}-3)\frac{\mathrm{x}}{2} + \cdots + \cos(\mathrm{n}-1)\frac{\mathrm{x}}{2}\right) =$$
$$\cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\,\frac{\left(\sin\frac{\mathrm{x}}{2}\cos(\mathrm{n}-1)\frac{\mathrm{x}}{2}+\sin\frac{\mathrm{x}}{2}\cos(\mathrm{n}-3)\frac{\mathrm{x}}{2}+\cdots+\sin\frac{\mathrm{x}}{2}\cos(\mathrm{n}-1)\frac{\mathrm{x}}{2}\right)}{\sin\frac{\mathrm{x}}{2}} =$$
$$\cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\,\frac{\frac{1}{2}\left(\sin\frac{\mathrm{nx}}{2}-\sin\frac{(\mathrm{n}-2)\mathrm{x}}{2}+\sin\frac{(\mathrm{n}-2)\mathrm{x}}{2}-\sin\frac{(\mathrm{n}-4)\mathrm{x}}{2}+\cdots+\sin\frac{\mathrm{nx}}{2}\right)}{\sin\frac{\mathrm{x}}{2}} = \cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\,\frac{\sin\frac{\mathrm{nx}}{2}}{\sin\frac{\mathrm{x}}{2}}.$$

It is interesting to look at a function $S\,(x)$.





Behavior of $S_1(x)$ is intuitively clear. For $x = 0$, all terms in the sum are equal to 1, and the sum equals to the number of terms, $S_1(0) = n$, while for $x \neq 0$ it consists of a large number of positive and negative terms, which tend to cancel each other.

2. Prove the following equalities:

   a. $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$

Solution:

$$\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \cot\frac{\alpha}{2}$$

   b. $\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$

Solution:

$$\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \left(\sin\left(\frac{7\pi}{8} - 2\alpha\right) - \sin\left(\frac{9\pi}{8} - 2\alpha\right)\right)\left(\sin\left(\frac{7\pi}{8} - 2\alpha\right) + \sin\left(\frac{9\pi}{8} - 2\alpha\right)\right) = 2\cos\frac{2\pi-4\alpha}{2}\sin\left(-\frac{\pi}{8}\right)2\sin\frac{2\pi-4\alpha}{2}\cos\left(-\frac{\pi}{8}\right) = -2\sin(\pi-2\alpha)\cos(\pi-2\alpha)\,2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) = \sin 4\alpha\sin\frac{\pi}{4} = \frac{\sin 4\alpha}{\sqrt{2}}$$

   c. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$

Solution:

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$$
$$=4\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}+4\cos^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}$$
$$=4\sin^2\frac{\alpha-\beta}{2}\left(\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right)=4\sin^2\frac{\alpha-\beta}{2}$$

d. $\frac{\cot^2 2\alpha-1}{2\cot 2\alpha}-\cos 8\alpha\cot 4\alpha=\sin 8\alpha$

Solution:

$\frac{\cot^2 2\alpha-1}{2\cot 2\alpha}-\cos 8\alpha\cot 4\alpha=\frac{\cos^2 2\alpha-\sin^2 2\alpha}{2\sin 2\alpha\cos 2\alpha}-\cos 8\alpha\frac{\cos 4\alpha}{\sin 4\alpha}=\cot 4\alpha\,(1-\cos 8\alpha)=\frac{\cos 4\alpha}{\sin 4\alpha}2\sin^2 4\alpha=2\sin 4\alpha\cos 4\alpha=\sin 8\alpha$

e. $\sin^6\alpha+\cos^6\alpha+3\sin^2\alpha\cos^2\alpha=1$

Solution:

$\sin^6\alpha+\cos^6\alpha+3\sin^2\alpha\cos^2\alpha=\left(\frac{3\sin\alpha-\sin 3\alpha}{4}\right)^2+\left(\frac{\cos 3\alpha+3\cos\alpha}{4}\right)^2+\frac{3}{4}\sin^2 2\alpha=\frac{1}{16}(9\sin^2\alpha+\sin^2 3\alpha-6\sin\alpha\sin 3\alpha+\cos^2 3\alpha+9\cos^2\alpha+6\cos 3\alpha\cos\alpha+12\sin^2 2\alpha)=\frac{1}{16}\big(10+6(\cos 3\alpha\cos\alpha-\sin 3\alpha\sin\alpha)+6(1-\cos 4\alpha)\big)=\frac{1}{16}\big(10+6\cos 4\alpha+6(1-\cos 4\alpha)\big)=1$

f. $\frac{\sin 6\alpha+\sin 7\alpha+\sin 8\alpha+\sin 9\alpha}{\cos 6\alpha+\cos 7\alpha+\cos 8\alpha+\cos 9\alpha}=\tan\frac{15\alpha}{2}$

Solution:

$$\frac{\sin 6\alpha+\sin 7\alpha+\sin 8\alpha+\sin 9\alpha}{\cos 6\alpha+\cos 7\alpha+\cos 8\alpha+\cos 9\alpha}=\frac{(\sin 6\alpha+\sin 9\alpha)+(\sin 8\alpha+\sin 7\alpha)}{(\cos 6\alpha+\cos 9\alpha)+(\cos 8\alpha+\cos 7\alpha)}=$$
$$\frac{2\sin\frac{15}{2}\alpha\cos\frac{3}{2}\alpha+2\sin\frac{15}{2}\alpha\cos\frac{1}{2}\alpha}{2\cos\frac{15}{2}\alpha\cos\frac{3}{2}\alpha+2\cos\frac{15}{2}\alpha\cos\frac{1}{2}\alpha}=\frac{\sin\frac{15}{2}\alpha}{\cos\frac{15}{2}\alpha}\frac{\cos\frac{3}{2}\alpha+\cos\frac{1}{2}\alpha}{\cos\frac{3}{2}\alpha+\cos\frac{1}{2}\alpha}=\tan\frac{15}{2}\alpha$$

g. $\sin^6\alpha+\cos^6\alpha=\frac{5+3\cos 4\alpha}{8}$

Solution:

$$\sin^6\alpha+\cos^6\alpha=\left(\frac{3\sin\alpha-\sin 3\alpha}{4}\right)^2+\left(\frac{\cos 3\alpha+3\cos\alpha}{4}\right)^2=\frac{1}{16}(9\sin^2\alpha+\sin^2 3\alpha-6\sin\alpha\sin 3\alpha+\cos^2 3\alpha+9\cos^2\alpha+6\cos 3\alpha\cos\alpha)=\frac{1}{16}\big(10+6(\cos 3\alpha\cos\alpha-\sin 3\alpha\sin\alpha)\big)=\frac{1}{16}(10+6\cos 4\alpha)=\frac{5+3\cos 4\alpha}{8}$$

h. $16\sin^5\alpha-20\sin^3\alpha+5\sin\alpha=\sin 5\alpha$

Solution:

$$\sin 5\alpha=\sin\alpha\cos 4\alpha+\cos\alpha\sin 4\alpha=\sin\alpha\,(2\cos^2 2\alpha-1)+\cos\alpha\, 2\sin 2\alpha\cos 2\alpha=\sin\alpha\,(2(1-2\sin^2\alpha)^2-1)+4\sin\alpha\cos^2\alpha\,(1-2\sin^2\alpha)=\sin\alpha\,\big(1-8\sin^2\alpha+8\sin^4\alpha+4(1-\sin^2\alpha)(1-2\sin^2\alpha)\big)=\sin\alpha\,(5-20\sin^2\alpha+16\sin^4\alpha)$$

i. $\frac{\cos 64^\circ\cos 4^\circ-\cos 86^\circ\cos 26^\circ}{\cos 71^\circ\cos 41^\circ-\cos 49^\circ\cos 19^\circ}$

Solution:

$$\frac{\cos 64^\circ\cos 4^\circ-\cos 86^\circ\cos 26^\circ}{\cos 71^\circ\cos 41^\circ-\cos 49^\circ\cos 19^\circ}=\frac{\cos 60^\circ+\cos 68^\circ-\cos 60^\circ-\cos 112^\circ}{\cos 30^\circ+\cos 112^\circ-\cos 30^\circ-\cos 68^\circ}=\frac{\cos 68^\circ-\cos 112^\circ}{\cos 112^\circ-\cos 68^\circ}=-1$$

j. $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=\frac{3}{16}$

Solution: denote $x=20^\circ$, $\cos 3x=\cos 60^\circ=\frac{1}{2}$, $\cos 6x=\cos 120^\circ=-\frac{1}{2}$,

$$\sin x\sin 2x\sin 3x\sin 4x=\sin x\sin 3x\sin 2x\sin 4x=\frac{1}{2}(\cos 2x-\cos 4x)\frac{1}{2}(\cos 2x-\cos 6x)=\frac{1}{2}\big(\cos 2x-(2\cos^2 2x-1)\big)\left(\cos 2x+\frac{1}{2}\right)=\frac{1}{2}\left(\cos^2 2x-2\cos^3 2x+\cos 2x+\frac{1}{2}\cos 2x-\cos^2 2x+\frac{1}{2}\right)=\frac{1}{4}(1-4\cos^3 2x+3\cos 2x)=\frac{1}{4}(1-\cos 6x)=\frac{3}{16}$$

k. $\frac{1}{\sin 10^\circ}-\frac{\sqrt{3}}{\cos 10^\circ}=4$

Solution:

$$\frac{1}{\sin 10^\circ}-\frac{\sqrt{3}}{\cos 10^\circ}=4\frac{\frac{1}{2}\cos 10^\circ-\frac{\sqrt{3}}{2}\sin 10^\circ}{2\sin 10^\circ\cos 10^\circ}=4\frac{\frac{1}{2}\cos 10^\circ-\frac{\sqrt{3}}{2}\sin 10^\circ}{2\sin 10^\circ\cos 10^\circ}=4\frac{\sin(30^\circ-10^\circ)}{\sin 20^\circ}=4$$

Trigonometry homework review. Part 2.

3. Simplify the following expressions:

   l. $\sin^2\left(\frac{\alpha}{2}+2\beta\right)-\sin^2\left(\frac{\alpha}{2}-2\beta\right)$

Solution:

$\sin^2\left(\frac{\alpha}{2}+2\beta\right)-\sin^2\left(\frac{\alpha}{2}-2\beta\right)=\left(\sin\left(\frac{\alpha}{2}+2\beta\right)-\sin\left(\frac{\alpha}{2}-2\beta\right)\right)\left(\sin\left(\frac{\alpha}{2}+2\beta\right)+\sin\left(\frac{\alpha}{2}-2\beta\right)\right)=2\cos\frac{\alpha}{2}\sin 2\beta\, 2\sin\frac{\alpha}{2}\cos 2\beta=\sin\alpha\sin 4\beta.$

   m. $2\cos^2 3\alpha+\sqrt{3}\sin 6\alpha-1$

Solution:

$$\begin{aligned}2\cos^2 3\alpha+\sqrt{3}\sin 6\alpha-1 &= \cos 6\alpha+1+\sqrt{3}\sin 6\alpha-1\\ &=2\left(\frac{1}{2}\cos 6\alpha+\frac{\sqrt{3}}{2}\sin 6\alpha\right)=2\left(\sin\frac{\pi}{6}\cos 6\alpha+\cos\frac{\pi}{6}\sin 6\alpha\right)\\ &=2\sin\left(\frac{\pi}{6}+6\alpha\right)\end{aligned}$$

   n. $\cos^4 2\alpha-6\cos^2 2\alpha\sin^2 2\alpha+\sin^4 2\alpha$

Solution:

$\cos^4 2\alpha-6\cos^2 2\alpha\sin^2 2\alpha+\sin^4 2\alpha=(\cos^2 2\alpha-\sin^2 2\alpha)^2-4\cos^2 2\alpha\sin^2 2\alpha=\cos^2 4\alpha-\sin^2 4\alpha=\cos 8\alpha$

   o. $\sin^2(135°-2\alpha)-\sin^2(210°-2\alpha)-\sin 195°\cos(165°-4\alpha).$

Solution:

$\sin^2(135°-2\alpha)-\sin^2(210°-2\alpha)-\sin\ 15°\cos(165°-4\alpha)=\sin^2(45°+2\alpha)-\sin^2(2\alpha-30°)-\sin\ 15°\cos(15°+4\alpha)=(\sin(45°+2\alpha)-\sin(2\alpha-30°))(\sin(45°+2\alpha)+\sin(2\alpha-30°))-\sin\ 15°\cos(15°+4\alpha)=2\cos\frac{15°+4\alpha}{2}\sin\frac{75°}{2}2\sin\frac{15°+4\alpha}{2}\cos\frac{75°}{2}-\sin\ 15°\cos(15°+4\alpha)=\sin 75°\sin(15°+4\alpha)-\sin\ 15°\cos(15°+4\alpha)=\sin(15°+4\alpha)\cos 15°-\cos(15°+4\alpha)\sin\ 15°=\sin(4\alpha)$

p. $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

Solution:

$$\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha} = \frac{2\sin 8\alpha \sin 6\alpha - 2\sin 8\alpha \sin 2\alpha}{2\sin 8\alpha \cos 2\alpha + 2\sin 8\alpha \cos 6\alpha}$$
$$= \frac{\sin 6\alpha - \sin 2\alpha}{\cos 2\alpha + \cos 6\alpha} = \frac{2\cos 4\alpha \sin 2\alpha}{2\cos 4\alpha \cos 2\alpha} = \tan 2\alpha$$

4. Let A, B and C be angles of a triangle. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

Solution:

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\left(\frac{\pi}{2} - \frac{A+B}{2}\right) + \tan\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\tan\frac{A}{2} = \tan\frac{A}{2}\tan\frac{B}{2} + \cot\frac{A+B}{2}\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = \tan\frac{A}{2}\tan\frac{B}{2} + \cot\frac{A+B}{2}\frac{\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \cot\frac{A+B}{2}\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin\frac{A}{2}\sin\frac{B}{2} + \cos\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = 1$$