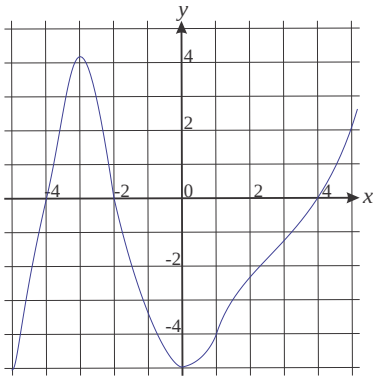
Homework for April 11, 2021.

## Algebra.

Review the classwork handout. Try solving the following problems. Remember: you do not necessarily need to solve all problems, just solve as many as you can within the time you can dedicate to Math 9 homework.

1. From the picture, find in which interval(s) the function $y = f(x)$
   a. is monotonic
   b. has the same sign




2. Find all possible values of $a$ such that equation $x^2 + ax + 9 = 0$ has two different roots, both of which are less than -1.
3. Draw graphs of the following functions
   a. $y = \left|\frac{1}{x-2} + 1\right|$
   b. $y = \frac{1}{|x|-2} + 1$
4. Solve the following equations
   a. (Skanavi 7.141) $3 \cdot 4^x + \frac{1}{3} \cdot 9^{x+2} = 6 \cdot 4^x - \frac{1}{2} \cdot 9^{x+1}$
   b. (Skanavi 7.143) $\sqrt{\log_x \sqrt{x}} = -\log_x 5$
   c. (Skanavi 7.153) $\frac{\log_2(9-2^x)}{3-x} = 1$
   d. (Skanavi 7.160) $\log_a x + \log_{a^2} x + \log_{a^3} x = 11$
   e. (Skanavi 7.184) $2^{x-1} + 2^{x-4} + 2^{x-2} = 6.5 + 3.25 + 1.625 + \cdots$
   f. (Skanavi 7.190) $9^x + 6^x = 2^{2x+1}$
   g. (Skanavi 7.197) $4^{\log x + 1} - 6^{\log x} - 2 \cdot 3^{\log x^2 + 2} = 0$
   h. (Skanavi 7.299) $(x^2 - x - 1)^{x^2-1} = 1$
   i. (Skanavi 7.304) find integer root: $\log_{\sqrt{x}}(x + 12) = 8 \log_{x+12} x$
   j. (Skanavi 7.308) $\log_{x+3}\left(3 - \sqrt{1 - 2x + x^2}\right) = \frac{1}{2}$
5. (Skanavi 7.277) Equation $4^x + 10^x = 25^x$ has a single root. Find this root. Is it positive or negative? Is it larger or less than 1?
6. (Skanavi 7.280) Show that:

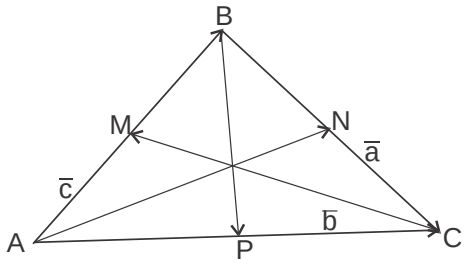
$$\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \log_7 6 \cdot \log_8 7 = \frac{1}{3}$$

## Geometry.

Review the classwork handout on vectors. Solve the following problems; skip the problems that were solved in class.

## Problems.

1. In a triangle ABC, vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ (**c**, **b** and **a**) are the sides. $\overrightarrow{AN}$, $\overrightarrow{CM}$ and $\overrightarrow{BP}$ are the medians.




   a. Express vectors $\overrightarrow{AN}$, $\overrightarrow{CM}$ and $\overrightarrow{BP}$ through vectors **c**, **b** and **a.**

   b. Find the sum of vectors $\overrightarrow{AN}$, $\overrightarrow{CM}$ and $\overrightarrow{BP}$.

2. Solve the same problem for bisectors $\overrightarrow{AN}$, $\overrightarrow{CM}$ and $\overrightarrow{BP}$ in a triangle $ABC$.

3. Coxeter, Greitzer, problem #9 to Sec. 2.1 (p. 31): How far away is the horizon as seen from the top of a mountain 1 mile high? (Assume the Earth to be a sphere of diameter 7920 miles.)

4. In a rectangle $ABCD$, $A_1$, $B_1$, $C_1$ and $D_1$ are the mid-points of sides $AB$, $CD$, $BC$ and $DA$, respectively. $M$ is the crossing point of the segments $A_1B_1$, and $C_1D_1$, connecting two pairs of midpoints.

   a. Express vector $\overrightarrow{A_1M}$ through $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$.

   b. Prove that $M$ is the mid-point of segments, $A_1B_1$ and $C_1D_1$, i.e. $|A_1M| = |MB_1|$ and $|C_1M| = |MD_1|$.

5. In a parallelogram $ABCD$, find $\overrightarrow{AB} + \overrightarrow{BD} - 2\overrightarrow{AD}$.

6. $M$ is a crossing point of the medians in a triangle $ABC$. Prove that $\overrightarrow{AM} = \frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.

7. For three points, $A(-1,3)$, $B(2,-5)$ and $C(3,4)$, find the (coordinates of) following vectors,

   a. $\overrightarrow{AB} - \overrightarrow{BC}$

   b. $\overrightarrow{AB} + \overrightarrow{CB} + \overrightarrow{AC}$

c. $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{CA}$

8. For two triangles, $ABC$ and $A_1B_1C_1$, $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 0$. Prove that medians of these two triangles cross at the same point $M$.

## Trigonometry exercise.

Review the trigonometry classwork handout. Solve the following problems. Some problems are repeated from previous trigonometry homeworks – skip those that you have already solved.

1. Find all $x$ for which,
   a. $\sin x \cos x = \frac{\sqrt{2}}{2}$
   b. $\sin x \cos x = \frac{1}{2}$
   c. $\sin x \cos x = \frac{\sqrt{3}}{4}$
2. Find the sum of the following series,

$$S = \cos x + \cos 3x + \cos 5x + \cos 7x + \cdots + \cos 2021x$$

(hint: multiply the sum by $2 \sin x$)

3. Calculate:
   a. $\cos 75° + \cos 15° =$
   b. $\cos\frac{\pi}{12} - \cos\frac{5\pi}{12} =$
4. Let $A$, $B$ and $C$ be angles of a triangle. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

5. Prove the following equalities:
   a. $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$
   b. $\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$
   c. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$
   d. $\frac{\cot^2 2\alpha - 1}{2\cot 2\alpha} - \cos 8\alpha \cot 4\alpha = \sin 8\alpha$

e. $\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha = 1$

f. $\frac{\sin 6\alpha+\sin 7\alpha+\sin 8\alpha+\sin 9\alpha}{\cos 6\alpha+\cos 7\alpha+\cos 8\alpha+\cos 9\alpha} = \tan\frac{15\alpha}{2}$

g. $\sin^6\alpha + \cos^6\alpha = \frac{5+3\cos 4\alpha}{8}$

h. $16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha = \sin 5\alpha$

i. $\frac{\cos 64^\circ\cos 4^\circ - \cos 86^\circ\cos 26^\circ}{\cos 71^\circ\cos 41^\circ - \cos 49^\circ\cos 19^\circ}$

j. $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ = \frac{3}{16}$

k. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$

6. Simplify the following expressions:

a. $\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right)$

b. $2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1$

c. $\cos^4 2\alpha - 6\cos^2 2\alpha\sin^2 2\alpha + \sin^4 2\alpha$

d. $\sin^2(135^\circ - 2\alpha) - \sin^2(210^\circ - 2\alpha) - \sin^2 195^\circ\cos(165^\circ - 4\alpha)$

e. $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

7. Solve the following equations and inequalities:

a. $\cos^2\pi x + 4\sin\pi x + 4 = 0$

b. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

c. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$

d. $\sin^2 x - 2\sin x\cos x = 3\cos^2 x$

e. $\sin 6x + 2 = 2\cos 4x$

f. $\cot x - \tan x = \sin x + \cos x$

g. $\sin x \geq \pi/2$

h. $\sin x \leq \cos x$