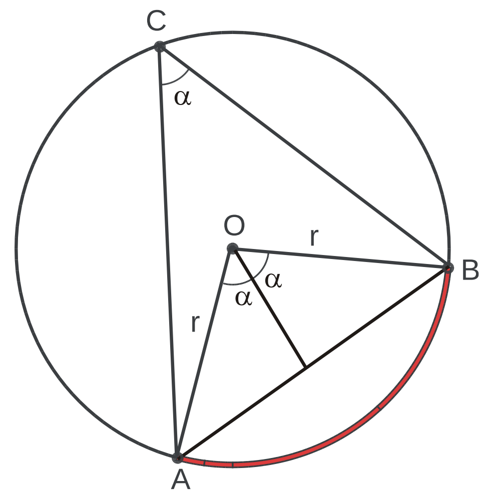
March 21, 2021 Math 9

Geometry.

Trigonometry homework review.

Problems.

1. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.



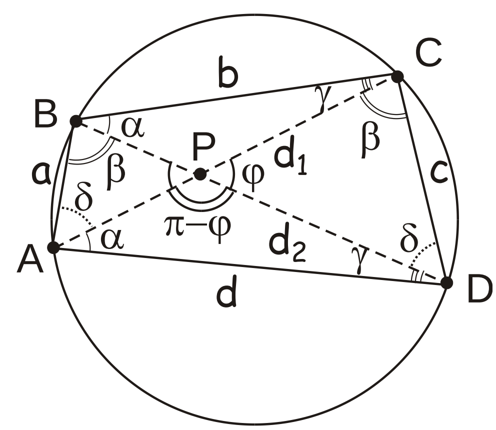
2. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):

   $\sin(\alpha+\beta)\sin(\beta+\gamma)=\sin\alpha\sin\gamma+\sin\beta\sin\delta$,

   if $\alpha+\beta+\gamma+\delta=\pi$.

3. Prove the Ptolemy identity in Problem 2 using the addition formulas for sine and cosine.

Solutions.



1. Consider the figure on the right,

$|AB|=2r\sin\alpha=\sin\alpha$, if $d=2r=1$.

2. According to Ptolemy's theorem for a quadrilateral inscribed in a circle,

$d_1d_2=ac+bd$.

Applying this for the circle of the unit diameter and using the result of the previous problem, we obtain,

$\sin(\alpha+\beta)\sin(\beta+\gamma)=\sin\alpha\sin\gamma+\sin\beta\sin\delta$, where $\alpha+\beta$ and $\gamma+\delta$ are the opposite angles of an inscribed quadrilateral (and so are $\alpha+\delta$ and $\beta+\gamma$), and therefore $\alpha+\beta+\gamma+\delta=\pi$.

3. Using the multiplication formulas for sines we obtain,

$\sin\alpha\sin\gamma+\sin\beta\sin\delta=\frac{1}{2}[\cos(\alpha-\gamma)-\cos(\alpha+\gamma)+\cos(\beta-\delta)-\cos(\beta+\delta)]=\frac{1}{2}\left[\cos(\alpha-\gamma)+\cos(\beta-\delta)-\left(\cos(\alpha+\gamma)+\cos(\beta+\delta)\right)\right]$

$= \frac{1}{2}\left[2\cos\left(\frac{\alpha-\gamma+\beta-\delta}{2}\right)\cos\left(\frac{\alpha-\gamma-\beta+\delta}{2}\right)\right] = \cos\left(\frac{2\alpha+2\beta-\pi}{2}\right)\cos\left(\frac{\pi-2\gamma-2\beta}{2}\right) =$
$\sin(\alpha+\beta)\sin(\beta+\gamma)$.

4. Using the Sine and the Cosine theorems, prove the Hero's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter.

Solution. The area of a triangle ABC is $S_{\Delta ABC} = \frac{1}{2}ab\sin\gamma$, so

$$\sin^2\gamma = \frac{4{S_{\Delta ABC}}^2}{a^2b^2}$$

From the Law of cosines, we have

$$\cos^2\gamma = \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$$

Adding the two expressions, we obtain, $1 = \frac{4{S_{\Delta ABC}}^2}{a^2b^2} + \frac{(a^2+b^2-c^2)^2}{4a^2b^2}$, or,

$16{S_{\Delta ABC}}^2 = 4a^2b^2 - (a^2+b^2-c^2)^2 = (2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2) = ((a+b)^2-c^2)(c^2-(a-b)^2)=(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$, or,

$${S_{\Delta ABC}}^2 = p(p-a)(p-b)(p-c)$$

5. Show that

a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \cos^2\alpha + \left(-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right)^2 + \left(-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right)^2 = \cos^2\alpha + 2\left(\frac{1}{2}\cos\alpha\right)^2 + 2\left(\frac{\sqrt{3}}{2}\sin\alpha\right)^2 = \frac{3}{2}\cos^2\alpha + \frac{3}{2}\sin^2\alpha = \frac{3}{2}$

b. $\sin\alpha + \sin\left(\frac{2\pi}{3} + \alpha\right) + \sin\left(\frac{4\pi}{3} + \alpha\right) = \sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha = 0$

c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{3\sin x - 4\sin^3 x}{\sin x} - \frac{4\cos^3 x - 3\cos x}{\cos x} = 6 - 4\sin^2 x - 4\cos^2 x = 2$

6. Without using calculator, find:

a. $\sin 75° = \sin(90° - 15°) = \cos 15° = \cos\frac{30°}{2} = \sqrt{\frac{1}{2}(1 + \cos 30°)} = \sqrt{\frac{2+\sqrt{3}}{4}}$

b. $\cos 75° = \cos(90° - 15°) = \sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1}{2}(1 - \cos 30°)} = \sqrt{\frac{2-\sqrt{3}}{4}}$

c. $\sin\frac{\pi}{8} = \sin\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2-\sqrt{2}}{4}}$

d. $\cos\frac{\pi}{8} = \cos\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2+\sqrt{2}}{4}}$

e. $\sin\frac{\pi}{16} = \sin\frac{1}{2}\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{8}\right)} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}}$

f. $\cos\frac{\pi}{16} = \cos\frac{1}{2}\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{8}\right)} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}}$

g. $\cos\frac{\pi}{2^{n+1}} = \cos\frac{1}{2}\left(\frac{\pi}{2^n}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{2^n}\right)} = \sqrt{\frac{2+\sqrt{2+\sqrt{2+\cdots}}}{4}}$

Trigonometric series.

Problem. Using the trigonometric formulas that we have previously derived, find the sum of the following trigonometric series,

$$S_N = \sin x + \sin 2x + \sin 3x + \sin 4x + \cdots + \sin Nx$$

Multiplying the sum with $2\sin\frac{x}{2}$ and using the formula for the product of two sines and then for the difference of two cosines we obtain,

$$2\sin\frac{x}{2}\cdot S_N = \cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{3x}{2} - \cos\frac{5x}{2} + \cdots + \cos\left(Nx - \frac{x}{2}\right)$$
$$- \cos\left(Nx + \frac{x}{2}\right) = \cos\frac{x}{2} - \cos\left(Nx + \frac{x}{2}\right) = 2\sin Nx \sin(N+1)x$$

Consequently,

$$S_N = \frac{\sin Nx \sin(N+1)x}{\sin\frac{x}{2}}$$

Vectors.

Recap on addition operation.

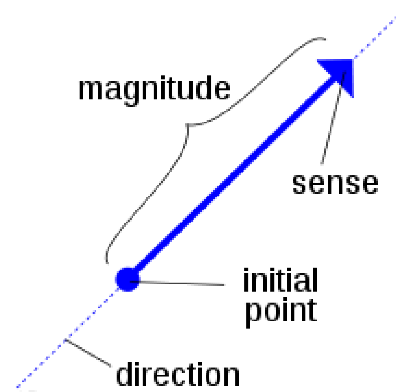
Addition is a mathematical operation that represents combining together objects, or collections of objects from a given set, into a new object, or collection. It is denoted by the plus sign (+). A nice review on various aspects of addition operation is on http://en.wikipedia.org/wiki/Addition.

Addition obeys several important laws. It is

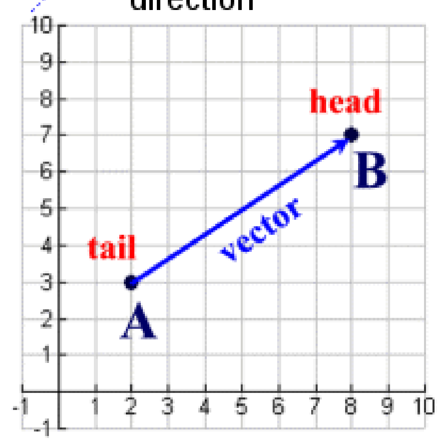
- **commutative**, meaning that order does not matter
- **associative**, for the number addition this means that when one adds more than two numbers, order in which addition is performed does not matter.
- addition of **zero**, denoted **0,** does not change the result; for numbers repeated addition of **1** is the same as counting.

Translations and vectors.

A **vector** is an object that has a magnitude (length) and a direction. This represents a **translation**. You may also hear the terms "displacement" vector or "translation" vector when working with translations.

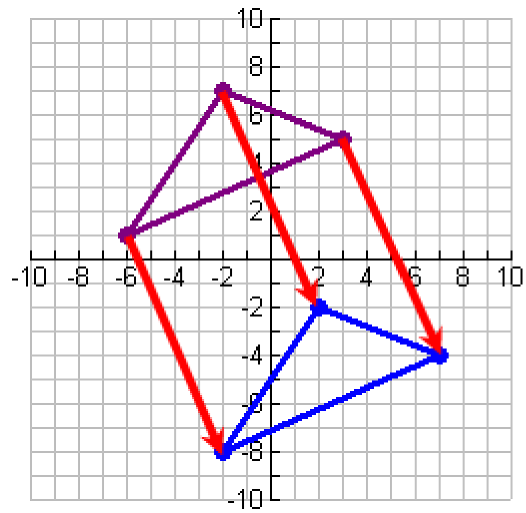


A vector is represented by a **directed line segment**, which is drawn as a segment with an arrow at one end indicating the direction of movement. Unlike a ray, which also has specific direction, a directed line segment has a specific length. The direction of a translation is indicated by an arrow pointing from the **tail** (the initial point) to the **head** (the terminal point).



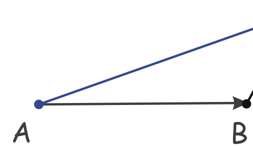
<u>Notation</u>. If the tail is at point A and the head is at point B, the vector from A to B is written as $\overrightarrow{AB}$. Vectors may also be labeled as a single letter with an arrow, $\overrightarrow{AB} = \vec{v}$, or a single bold face letter, such as vector **v**. The **length** (magnitude) of a vector **v** is denoted **|v|**. Length is always a non-negative real number.

The translation at the right shows a vector translating the top triangle 4 units to the right and 9 units downward. The notation for such vector movement may be written as: $(4, -9)$, or $\begin{pmatrix} 4 \\ -9 \end{pmatrix}$. Vectors such as those used in a given translation form an infinite set of <u>parallel directed line segments and can be thought of as a translation</u>. Any two vectors of the same length and parallel to each other are considered identical. They need not have the same initial and terminal points. Notice that the vectors in the translation which connect the pre-image vertices to the image vertices are all parallel and are all the same length. In other words, a vector can be identified with the equivalence class of a directed line segment specifying given translation.

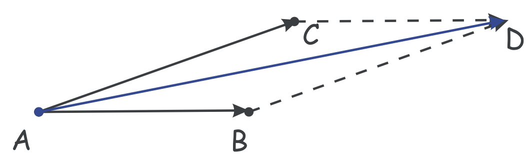


Addition (subtraction) of vectors.

One can formally define an operation of addition on the set of all vectors (in the space of vectors). For any two vectors, $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{CD} = \vec{b}$, such an operation results in a third vector, $\overrightarrow{EF} = \vec{c}$, such that three following rules hold,



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$
$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$$



$$\vec{a} + \vec{b} = \vec{b} + \vec{a} = \vec{c} \qquad (1)$$

$$\forall\ \overrightarrow{MN} = \vec{d},\ \left(\vec{a} + \vec{b}\right) + \vec{d} = \vec{a} + \left(\vec{b} + \vec{d}\right) \qquad (2)$$

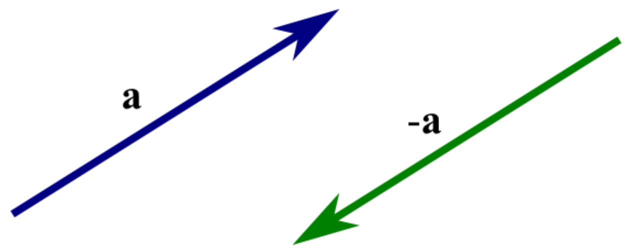
$$\exists\ \vec{0},\ \forall \vec{a}\ ,\ \vec{a} + \vec{0} = \vec{a} \qquad (3)$$

Usually (but not necessarily always), vector addition also satisfies the fourth property,

$$\forall \vec{a}\ ,\ \exists\ -\vec{a},\ \vec{a} + (-\vec{a}) = \vec{0}, \qquad (4)$$

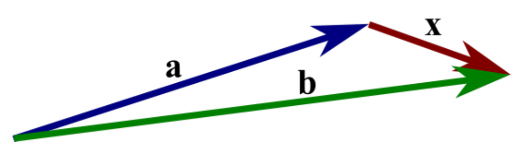
which allows to define the subtraction of vectors (this is not true for the so-called “pseudo-vectors”, such as representing rotations instead of translations; rotations do not satisfy commutativity (1) either).

The vector $-\vec{a}$ is the vector with the same magnitude as $\vec{a}$ but pointing in the opposite direction. We define the subtraction as an addition of this opposite-pointing vector:



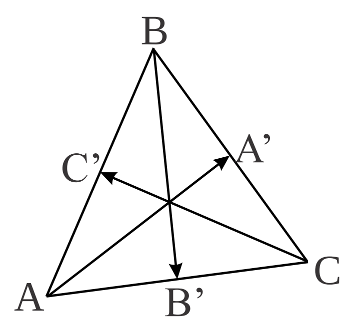
$$\vec{b} - \vec{a} = \vec{b} + (-\vec{a})$$

Can you see how the vector $\vec{x}$ in the figure below is equal to $\vec{b} - \vec{a}$? Notice how this is the same as stating that $\vec{a} + \vec{x} = \vec{b}$**,** similar to subtraction of scalar numbers.



Addition and subtraction of vectors can be simply illustrated by considering the consecutive translations. They allow decomposing any vector $\vec{a}$ into a sum, or a difference, of a number (two, or more) of other vectors.

Exercise.



1. $\boldsymbol{AA'}$, $\boldsymbol{BB'}$ and $\boldsymbol{CC'}$ are the medians of the triangle $\boldsymbol{ABC}$. Prove that

$$\overrightarrow{\boldsymbol{AA'}} + \overrightarrow{\boldsymbol{BB'}} + \overrightarrow{\boldsymbol{CC'}} = \boldsymbol{0}$$

2. Prove that if for vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$**,**

$$|\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}| = |\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}|$$

then $\vec{\boldsymbol{a}} \perp \vec{\boldsymbol{b}}$.

Multiplication with a scalar

Given a vector $\vec{a}$ and a real number (scalar) λ, we can define vector $\lambda\vec{a}$ as follows. If λ is positive, then $\lambda\vec{a}$ is the vector whose direction is the same as the direction of $\vec{a}$ and whose length is λ times the length of $\vec{a}$. In this case, multiplication by λ simply stretches (if $\lambda > 1$) or compresses (if $0 < \lambda < 1$) the vector $\vec{a}$.

If, on the other hand, λ is negative, then we have to take the opposite of $\vec{a}$ before stretching or compressing it. In other words, the vector $\lambda\vec{a}$ points in the opposite direction of $\vec{a}$, and the length of $\lambda\vec{a}$ is $|\lambda|$ times the length of $\vec{a}$. No matter the sign of λ, we observe that the magnitude of $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of $\vec{a}$, $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

Multiplication with a scalar satisfies many of the same properties as the usual multiplication.

1. $s \cdot (\vec{a} + \vec{b}) = s\vec{a} + s\vec{b}$ (distributive law, form 1)

2. $(s + t) \cdot \vec{a} = s\vec{a} + t\vec{a}$ (distributive law, form 2)

3. $1 \cdot \vec{a} = \vec{a}$

4. $(-1) \cdot \vec{a} = -\vec{a}$

5. $0 \cdot \vec{a} = \vec{0}$

In the last formula, the zero on the left is the number 0, while the zero on the right is the vector $\vec{0}$, which is the unique vector whose length is zero.

If $\vec{a} = \lambda\vec{b}$ for some scalar λ, then we say that the vectors $\vec{a}$ and $\vec{b}$ are parallel. If λ is negative, some people say that $\vec{a}$ and $\vec{b}$ are anti-parallel.

Scalar (dot) product of vectors.

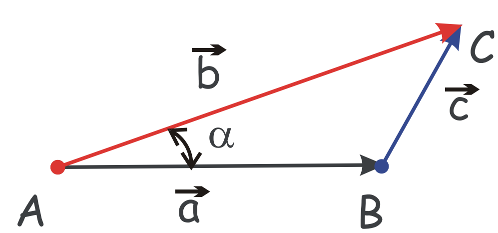
One can formally define an operation of scalar multiplication on vectors, consistent with the following definition of length, or magnitude of a vector,

$$(\vec{c} \cdot \vec{c}) = (\vec{c}\,)^2 = c^2. \qquad (8)$$

and the following properties, which hold if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are **real vectors** and r is a **scalar**.

- The dot product is **commutative**:

  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$



- The dot product is **distributive** over vector addition:

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

- The dot product is **bilinear**:

$$\vec{a} \cdot (r\vec{b} + \vec{c}) = r(\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})$$

$$\vec{a} - \vec{b} = \vec{c}$$
$$c^2 = (\vec{a} - \vec{b})^2 = a^2 + b^2 - 2(\vec{a} \cdot \vec{b})$$

- When multiplied by a scalar value, dot product satisfies:

$$(c_1\vec{a}) \cdot (c_2\vec{b}) = (c_1 c_2)(\overrightarrow{a \cdot b})$$

(these last two properties follow from the first two). Then, it is clear from the drawing at the right, that

$$\vec{a} \cdot \vec{b} = ab \cos\left(\widehat{\vec{a}, \vec{b}}\right) = ab \cos \alpha. \qquad (9)$$

Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **orthogonal** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Unlike multiplication of ordinary numbers, where if $ab = ac$, then b always equals c unless a is zero, the dot product **does not obey the cancellation law**:

If $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c}$ and $\boldsymbol{a} \neq \boldsymbol{0}$, then we can write: $\boldsymbol{a} \cdot (\boldsymbol{b} - \boldsymbol{c}) = 0$ by the distributive law; the result above says this just means that $\boldsymbol{a}$ is perpendicular to $(\boldsymbol{b} - \boldsymbol{c})$, which still allows $(\boldsymbol{b} - \boldsymbol{c}) \neq \boldsymbol{0}$, and therefore $\boldsymbol{b} \neq \boldsymbol{c}$.

The coordinate representation of vectors.

Let vector $\boldsymbol{a}$ define translation $A \to B$ under which an arbitrary point A with coordinates (x_A, y_A) on the (X, Y) coordinate plane is displaced to a point B with coordinates $(x_B, y_B) = (x_A + a_x, y_A + a_y)$ on this plane. Then, vector $\boldsymbol{a}$ is fully determined by the pair of numbers, (a_x, a_y), which specify displacements along X and Y axis, respectively. If O is the point of origin in the coordinate plane, $(x_O, y_O) = (0, 0)$, it is clear, that

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \vec{a}, \qquad (5)$$

from where follows the coordinate notation of vectors, $\overrightarrow{OA} = (x_A, y_A)$, $\overrightarrow{OB} = (x_B, y_B)$, $\vec{a} = (a_x, a_y) = \boldsymbol{a}$. Since numbers a_x and a_y denote magnitudes of translation along the X and Y axes, respectively, corresponding to the displacement by a vector $\boldsymbol{a}$, it can be represented as a sum of these two translations, $\boldsymbol{a} = a_x \, \boldsymbol{e}_x + a_y \, \boldsymbol{e}_y$, or, $(a_x, a_y) = (a_x, 0) + (0, a_y)$, or,

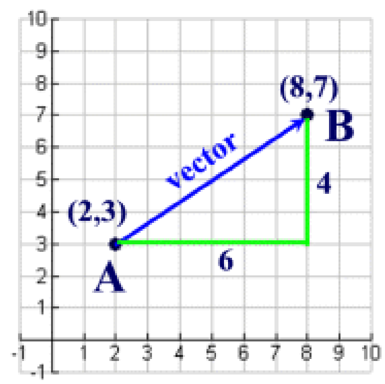
$$\vec{a} = a_x \overrightarrow{e_x} + a_y \overrightarrow{e_y}, \qquad (6)$$

where $\boldsymbol{e}_x = \overrightarrow{e_x}$ and $\boldsymbol{e}_y = \overrightarrow{e_y}$ are vectors of length 1, called unit vectors.

The length (magnitude) of a vector, in coordinate representation, is

$$|\overrightarrow{OA}| = \sqrt{x_A^2 + y_A^2}, \text{ a} = |\mathbf{a}| = |\vec{a}| = \sqrt{a_x^2 + a_y^2}. \qquad (7)$$

As you can see in the diagram at the right, the length of a vector can be found by forming a right triangle and utilizing the Pythagorean Theorem or by using the Distance Formula.

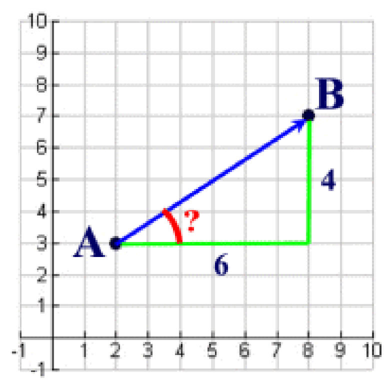


The vector at the right translates 6 units to the right and 4 units upward. The magnitude of the vector is $2\sqrt{13}$ from the Pythagorean Theorem, or from the Distance Formula:

$$|AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

The direction of a vector is determined by the angle it makes with a horizontal line.

In the diagram at the right, to find the direction of the vector (in degrees) we will utilize trigonometry. The tangent of the angle formed by the vector and the horizontal line (the one drawn parallel to the *x*-axis) is 4/6 (opposite/adjacent).



$$\tan \hat{A} = \frac{4}{6} \Rightarrow \hat{A} = \tan^{-1}\frac{4}{6} \approx 33.7°$$

The vector operations are easy to express in terms of these coordinates. If $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$, their sum is simply

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y),$$

as illustrated in the figure below. It is also easy to see that

$$\vec{b} - \vec{a} = (b_x - a_x, b_y - a_y)$$

Using the coordinate representation, we can define multiplication of a vector with a (scalar) number λ,

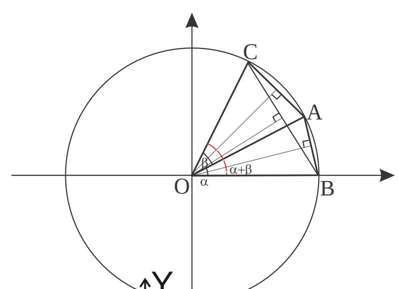
$$\lambda\vec{a} = (\lambda a_x, \lambda a_y)$$

Multiplication of a vector with a scalar satisfies the distributive law,

$$\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b} = \big(\lambda(a_x + b_x), \lambda(a_y + b_y)\big)$$

Coordinate representation of vectors. The dot product.

The coordinate representation for the dot product is $(\boldsymbol{a} \cdot \boldsymbol{b}) = a_x b_x + a_y b_y$. According to the definition given above, this must be equivalent to $(\boldsymbol{a} \cdot \boldsymbol{b}) = ab\ \cos(\widehat{\boldsymbol{ab}})$, which could be straightforwardly verified from the drawing on the right.



This equivalence follows immediately from the formula for the cosine of a difference of two angles,

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, therefore,

$(\overrightarrow{OA} \cdot \overrightarrow{OB}) = |OA||OB|\cos(\alpha - \beta) =$
$|OA|\cos\alpha\,|OB|\cos\beta + |OA|\sin\alpha\,|OB|\sin\beta =$
$a_x\, b_x + a_y\, b_y.$