Homework for March 21, 2021.

Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework. Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that "catch your eye", in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past.

1. Let x_1, x_2 and x_3 be distinct real numbers. Prove that there exists a unique polynomial, $P(x)$, of degree 2 such that $P(x_1) = 1, P(x_2) = P(x_3) = 0$. [Hint: if $P(x_1) = 0$, then $P(x)$ is divisible by $(x - x_1)$.] Find this polynomial if $x_1 = 2, x_2 = -1, x_3 = 5$.
2. As before, let x_1, x_2 and x_3 be distinct real numbers, and let y_1, y_2 and y_3 be any collection of numbers. Prove that there is a unique quadratic polynomial $f(x)$ such that $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3$. Find this polynomial if $x_1 = 2, x_2 = -1, x_3 = 5, y_1 = 3, y_2 = 6, y_3 = 18$. [Hint: look for in the form $f(x) = y_1 f(x_1) + \cdots$.]
3. Prove the following general result: given numbers $x_1, \dots, x_n, y_1, \dots, y_n$, such that x_i are distinct, there exists a unique polynomial $f(x)$ of degree $n - 1$ such that $f(x_i) = y_i, i = 1, \dots, n$. (For $n = 2$, this is a statement that there is a unique line through two given points.)
4. Prove that if $P(x)$ is a polynomial with integer coefficients, then for any integer a, b, the difference $P(a) - P(b)$ is divisible by $a - b$.
5. Le t x_1 and x_2 be the roots of the polynomial, $x^2 + 7x - 3$. Find
   a. ${x_1}^2 + {x_2}^2$
   b. $\frac{1}{x_1} + \frac{1}{x_2}$
   c. $(x_1 - x_2)^2$
   d. ${x_1}^3 + {x_2}^3$
6. What is the maximum number of different integer roots that the following polynomial can have? How the answer changes if $a = 0$?

$$x^{10} + ax^9 + bx^8 + cx^7 + dx^6 + fx^5 + gx^4 + hx^3 + kx^2 + lx^2 + mx = 1024$$

Geometry/Trigonometry.

Read the classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Additional reading on trigonometric functions is Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163),
http://en.wikipedia.org/wiki/Trigonometric_functions
http://en.wikipedia.org/wiki/Sine. Solve the following problems.

1. Using the expressions for the sine and the cosine of the sum of two angles derived in class, derive expressions for:
   a. $\sin 3\alpha$
   b. $\cos 3\alpha$
   c. $\tan(\alpha \pm \beta)$
   d. $\cot(\alpha \pm \beta)$
   e. $\tan(2\alpha)$
   f. $\cot(2\alpha)$
2. Derive the following expressions (classwork exercise):
   a. $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha)\cos(\beta)}$ $\quad \cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha\pm\beta)}{\sin(\alpha)\sin(\beta)}$
   b. $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$ $\quad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$
   c. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
   d. $\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$ $\quad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$
   e. $\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1-\cos\alpha)}$ $\quad \cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1+\cos\alpha)}$
   f. $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$
   g. $\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\sin\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$
   h. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$
   i. $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$
   j. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$
3. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.

4. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):

$$\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta,$$

if $\alpha+\beta+\gamma+\delta = \pi$.

5. Prove the Ptolemy identity in Problem 2 using the addition formulas for sine and cosine.

6. Using the Sine and the Cosine theorems, prove the Heron's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter.

7. Show that

a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \frac{3}{2}$

b. $\sin\alpha + \sin\left(\frac{2\pi}{3}+\alpha\right) + \sin\left(\frac{4\pi}{3}+\alpha\right) = 0$

c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

8. Without using calculator, find:

a. $\sin 75° =$

b. $\cos 75° =$

c. $\sin\frac{\pi}{8} =$

d. $\cos\frac{\pi}{8} =$

e. $\sin\frac{\pi}{16} =$

f. $\cos\frac{\pi}{16} =$