Homework for March 7, 2021.

Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework (some are repeated below). Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that “catch your eye”, in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past.

1. Perform long division of the following polynomials.
   a. $(x^5 - 2x^3 + 3x^2 - 4) \div (x^2 - x + 1)$
   b. $(x^4 - x^2 + 1) \div (x + 1)$
   c. $(x^7 + 1) \div (x^3 - x + 1)$
   d. $(6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x - 1) \div (x^2 + 1)$
   e. $(x^5 - 32) \div (x + 2)$
   f. $(x^5 - 32) \div (x - 2)$
   g. $(x^6 + 64) \div (x^2 + 4)$
   h. $(x^6 + 64) \div (x^2 - 4)$
   i. $(x^{100} - 1) \div (x^2 - 1)$
2. Can you find coefficients a, b, such that there is no remainder upon division of a polynomial, $x^4 + ax^3 + bx^2 - 2x - 10$,
   a. by $x + 5$
   b. by $x^2 + x - 1$
3. Prove that,
   a. for odd n, the polynomial $x^n + 1$ is divisible by $x + 1$
   b. $2^{100} + 1$ is divisible by 17.
   c. $2^n + 1$ can only be prime if n is a power of 2 [Primes of this form are called Fermat primes; there are very few of them. How many can you find?]
   d. for any natural number n, $8^n - 1$ is divisible by 7.
   e. for any natural number n, $15^n + 6$ is divisible by 7
4. Factor (i.e., write as a product of polynomials of smaller degree) the following polynomials.

   a. $1 + a + a^2 + a^3$
   b. $1 - a + a^2 - a^3 + a^4 - a^5$
   c. $a^3 + 3a^2b + 3b^2a + b^3$

d. $x^4 - 3x^2 + 2$

5. Simplify the following expressions using polynomial factorization.
   e. $\frac{x+y}{x} - \frac{x}{x-y} + \frac{y^2}{x^2-xy}$
   f. $\frac{x^6-1}{x^4+x^2+1}$
   g. $\frac{a^3-2a^2+5a+26}{a^3-5a^2+17a-13}$
6. Solve the following equations
   h. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = 2.9$ (hint: substitution)
   i. $\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0$ (hint: factorize square polynomials)
7. Write Vieta formulae for the reduced cubic equation, $x^3 + px + q = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of p and q,
   j. $(x_1 + x_2 + x_3)^2$
   k. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   l. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   m. $(x_1 + x_2 + x_3)^3$
8. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 7$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{7}$$

   Prove that then, at least one of x, y, z is equal to 7. [Hint: Vieta formulas]

9. Find all real roots of the following polynomial and factor it: $x^4 - x^3 + 5x^2 - x - 6$.
10. Prove that the following numbers are irrational:
    a. $\sqrt[2]{2}$
    b. $\sqrt[3]{3}$
    c. $\sqrt[5]{5}$
11. Compare the following real numbers (are they equal? which is larger?)

a. 1.33333… = 1.(3) and 4/3
b. 0.09999… = 0.0(9) and 1/10
c. 99.9999… = 99.(9) and 100
d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$

Geometry.

Read the classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Additional reading on trigonometric functions is Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163), http://en.wikipedia.org/wiki/Trigonometric_functions http://en.wikipedia.org/wiki/Sine. Solve the following problems.

Problems.

1. Find the distance between the nearest points of the circles,
   a. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 9$
   b. $(x+3)^2 + y^2 = 4$ and $x^2 + (y-4)^2 = 9$
   c. $(x-2)^2 + (y+1)^2 = 4$ and $(x+1)^2 + (y-3)^2 = 5$
   d. $(x-a)^2 + y^2 = r_1^2$ and $x^2 + (y-b)^2 = r_2^2$
2. Review derivation of the equation describing an ellipse and derive in a similar way,
   a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
   b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
3. Find (describe) set of all points formed by the centers of the circles that are tangent to a given circle of radius r and a line at a distance $d > r$ from its center, O.
4. Using the expressions for the sine and the cosine of the sum of two angles derived in class, derive expressions for (classwork exercise),
   a. $\sin 3\alpha$

b. $\cos 3\alpha$
c. $\tan(\alpha \pm \beta)$
d. $\cot(\alpha \pm \beta)$
e. $\tan(2\alpha)$
f. $\cot(2\alpha)$