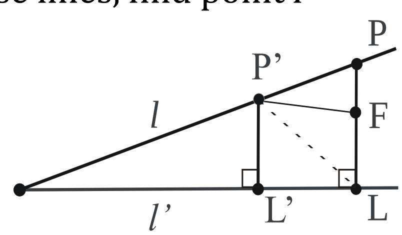
Homework for February 28, 2021.

Geometry.

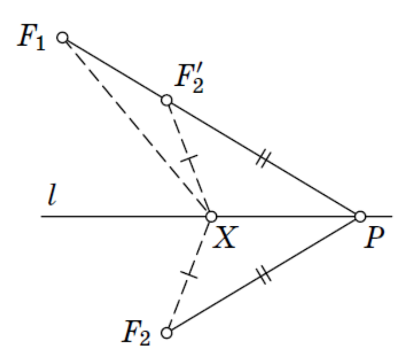
Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework (some are repeated below – skip the ones you have already done).

Problems.

1. Review derivation of the equation describing an ellipse and derive in a similar way,
    a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
    b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Given two lines, l and l', and a point F not on any of those lines, find point P on l such that the (signed) difference of distances from it to l' and F, $|P'L'| - |P'F|$, is maximal. As seen in the figure, for any P' on l the distance to l', $|P'L'| \leq |P'L| \leq |P'F| + |FL|$, where $|FL|$ is the distance from F to l'. Hence, $|P'L'| - |P'F| \leq |FL|$, and the difference is largest ($= |FL|$) when point P belongs to the perpendicular FL from point F to l'.

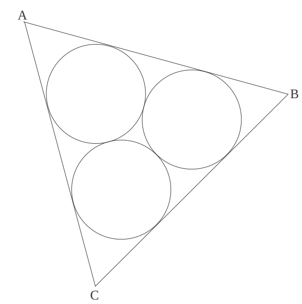


3. Given line l and points F_1 and F_2 lying on different sides of it, find point P on the line l such that the absolute value of the difference in distances from P to points F_1 and F_2 is maximal. As above, let $F_2{}'$ be the reflection of F_2 in l. Then for any point X on l, $|XF_2| - |XF_1'| \leq |F_1F_2'|$.



4. Find the (x, y) coordinates of the common (intersection) point of the two lines, one passing through the origin at 45 degrees to the X-axis, and the other passing through the point (1,0) at 60 degrees to it.

5. Find the (x, y) coordinates of the common (intersection) points of the parabola $y = x^2$ and of the ellipse centered at the origin and with major axis along the Y-axis whose length equals 2, and the minor axis along the X-axis whose length equals 1.
6. (Skanavi 10.122) Find the locus of the midpoints of all chords of a given circle with the center O, which intersect given chord AB of this circle.
7. Three circles of radius r touch each other. Find the area of the triangle ABC formed by tangents to pairs of circles (see figure).



8. Consider all possible configurations of the Apollonius problem (i.e. different possible choices of circles, points and lines). How many possibilities are there? Make the corresponding drawings and write the equations for finding the Apollonius circle in one of them (of your choice).
9. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where a, b, m, n are real numbers.
10. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.

Algebra.

Review the classwork handout and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
    a. 1/8
    b. 1/7
    c. 2/7
    d. 1/6
    e. 1/15
    f. 1/14
    g. 0.1
    h. 0.33333... = 0.(3)
    i. 0.13333... = 0.1(3)

2. Prove the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

a. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
   i. $a = b$
   ii. $a < b$
   iii. $a > b$
b. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
c. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
d. Archimedean property: $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}: a < nb$

Addition and subtraction.

a. $\forall\, a, b \in \mathbb{R}, a + b = b + a$
b. $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
c. $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
d. $\forall\, a \in \mathbb{R}, \exists - a \in \mathbb{R}, a + (-a) = 0$
e. $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
f. $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

2. Show that for the set of real numbers, $\mathbb{R}$, cardinality of the set of all possible subsets is greater than that of a continuum of real numbers itself.
3. Prove that the cardinality of the set of all points on a sphere is the same as that of the set of all points on a circle.
4. Represent $\sqrt{2}$ (and $\sqrt{p}$ for any rational p) by using the continuous fraction,

$$\sqrt{2} = a + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b + \cdots}}}$$