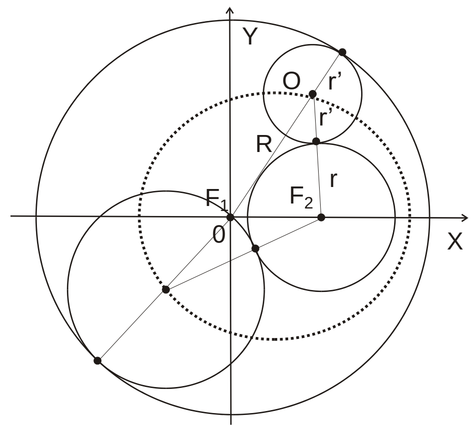
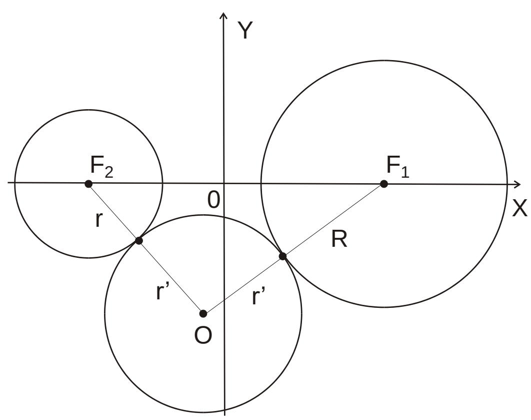
Geometry.

Ellipse. Hyperbola. Parabola (continued).

Alternate definitions of ellipse, hyperbola and parabola: Tangent circles.



Ellipse is the locus of centers of all circles tangent to two given nested circles (F_1, R) and (F_2, r). Its foci are the centers of these given circles, F_1 and F_2, and the major axis equals the sum of the radii of the two circles, $2a = R + r$ (if circles are externally tangential to both given circles, as shown in the figure), or the difference of their radii (if circles contain smaller circle (F_2, r).).



Consider circles (F_1, R) and (F_2, r). that are not nested. Then the loci of the centers O of circles externally tangent to these two satisfy $|OF_1| - |O\,F_2| = R - r$.

Hyperbola is the locus of the centers of circles tangent to two given non-nested circles. Its foci are the centers of these given circles, and the vertex distance $2a$ equals the difference in radii of the two circles.

As a special case, one given circle may be a point located at one focus; since a point may be considered as a circle of zero radius, the other given circle—which is centered on the other focus—must have radius $2a$. This provides a simple technique for constructing a hyperbola. It follows from this definition that a tangent line to the hyperbola at a point P bisects the angle formed with the two foci, i.e., the angle F_1PF_2. Consequently, the feet of perpendiculars drawn from each focus to such a tangent line lies on a circle of radius a that is centered on the hyperbola's own center.

If the radius of one of the given circles is zero, then it shrinks to a point, and if the radius of the other given circle becomes infinitely large, then the "circle" becomes just a straight line.

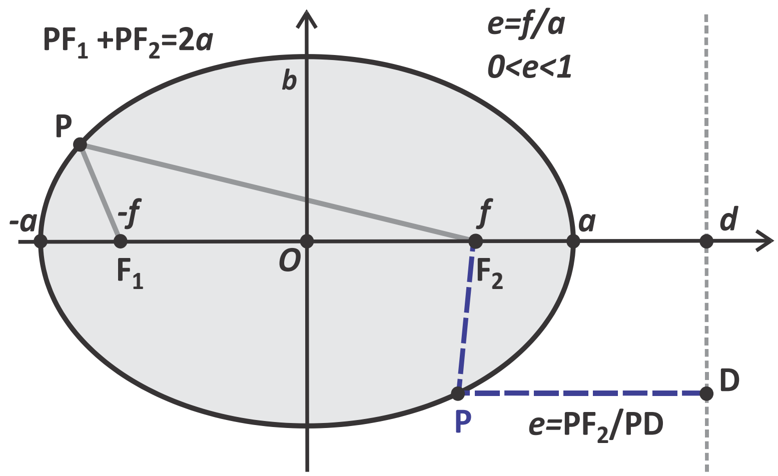
<u>Parabola</u> is the locus of the centers of circles passing through a given point and tangent to a given line. The point is the focus of the parabola, and the line is the directrix.

Alternate definitions of ellipse, hyperbola and parabola: Directrix and Focus.

<u>Parabola</u> is the locus of points such that the ratio of the distance to a given point (focus) and a given line (directrix) equals 1.

<u>Ellipse</u> can be defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.

<u>Hyperbola</u> can also be defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$. This constant is the eccentricity of the hyperbola. By symmetry a hyperbola has two directrices, which are parallel to the conjugate axis and are between it and the tangent to the hyperbola at a vertex.



In order to show that the above definitions indeed those of an ellipse and a hyperbola, let us obtain relation between the x and y coordinates of a point P (x, y) satisfying the definition. Using axes shown in the Figure, with focus F_2 on the X axis at a distance l from the origin and choosing the Y-axis for the directrix, we have

$$\frac{\sqrt{(x-l)^2 + y^2}}{x} = e$$

$$(x-l)^2 + y^2 = (ex)^2$$

$$x^2(1-e^2)\ -2lx+l^2+y^2=0$$

$$(1-e^2)\left(x^2-2x\frac{l}{1-e^2}+\left(\frac{l}{1-e^2}\right)^2\right)+y^2=\frac{l^2}{1-e^2}-l^2=\frac{e^2l^2}{1-e^2}$$

Finally, we thus obtain,

$$\frac{(x-\frac{l}{1-e^2})^2}{\frac{e^2l^2}{(1-e^2)^2}}+\frac{y^2}{\frac{e^2l^2}{1-e^2}}=1$$

Which is the equation of an ellipse for $1-e^2>0$ and of a hyperbola for $1-e^2<0$. In each case the center is at $x=x_0=\frac{l}{1-e^2}$ and $y=y_0=0$, and the semi-axes are $a=\frac{e\ l}{(1-e^2)}$ and $b=\frac{e\ l}{\sqrt{|1-e^2|}}$, which brings the equation to a canonical form,

$$\frac{(x-x_0)^2}{a^2}\pm\frac{(y-y_0)^2}{b^2}=1$$

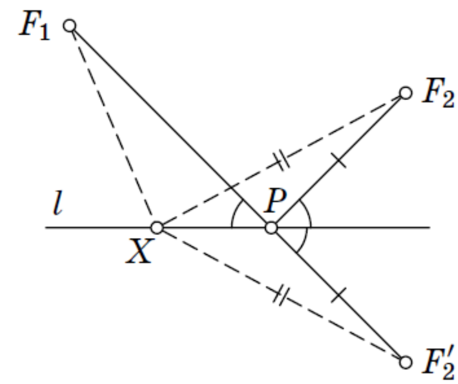
We also obtain the following relations between the eccentricity e and the ratio of the semi-axes, a/b: $\frac{b}{a}=\sqrt{|1-e^2|}$, or, $e=\sqrt{1\pm\left(\frac{b}{a}\right)^2}$, where plus and minus sign correspond to the case of a hyperbola and an ellipse, respectively.

Curves of the second degree.

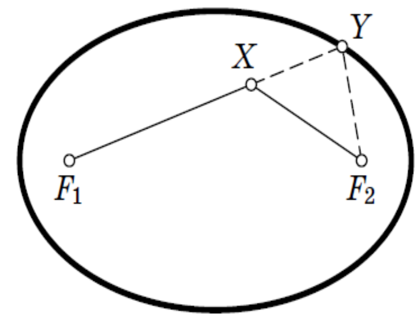
<u>A curve of the second degree</u> is a set of points whose coordinates in some (and therefore in any) Cartesian coordinate system satisfy a second order equation,

$$a_{11}x^2+a_{12}xy+a_{22}x^2+2b_1x+2b_2y+c=0$$

Curves of the second degree. The optical property.

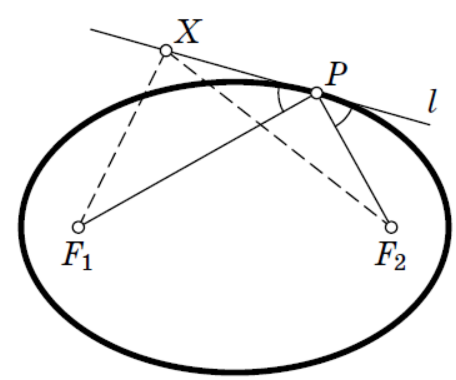


<u>Fermat principle and the mirror reflection</u>. If a ray of light is reflected in a mirror, then the reflection angle equals the incidence angle. This follows from the <u>Fermat principle</u>, which states that the light always travels along the shortest path. It is clear from the Figure that of all reflection points P on the line l (mirror) the shortest path between points F_1 and F_2 on the same side of it is such that points F_1, P, and the reflection of F_2 in l, $F_2{}'$, lie on a straight line.



<u>The interior and exterior points of an ellipse</u>. The sum of the distances from any point inside the ellipse to the foci is less, and from any point outside the ellipse is greater, than the length of the major axis.

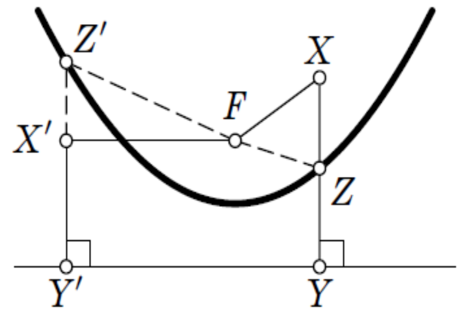
<u>Proof</u>. Let X be a point inside an ellipse with foci F_1, F_2. Using the triangle inequality, $|XF_2| < |XY| + |YF_2|$, we obtain, $|F_1X| + |XF_2| < |F_1X| + |XY| + |YF_2| = |F_1Y| + |YF_2|$. Similarly, if X is outside an ellipse, $|F_1X| + |XF_2| = |F_1X| + |XY| + |YF_2| > |F_1Y| + |YF_2|$.



<u>The optical property of the ellipse</u>. A light ray passing through one focus of an elliptical mirror will pass through another focus upon reflection.

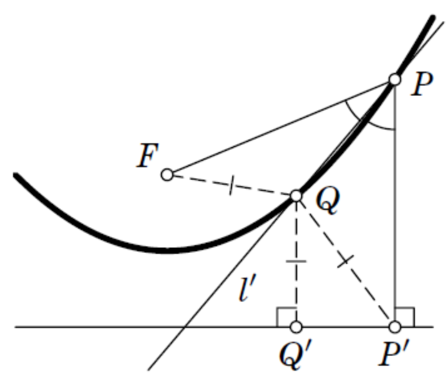
<u>Proof</u>. Suppose a line l is tangent to an ellipse at a point P. Then, l is the bisector of the exterior angle F_1PF_2 (and its perpendicular at point P is the bisector of F_1PF_2). Let X be an arbitrary point of l different from P. Since X is outside the ellipse, we have $|F_1X| + |XF_2| > |F_1P| + |PF_2|$, i.e., of all the points of l the point P has the smallest sum of the distances to F_1 and F_2. This means that the angles formed by the lines PF_1 and PF_2 with l are equal.

The interior and exterior points of a parabola. For the points inside a parabola the distance to the focus is less than the distance to the directrix, and for the points outside the parabola the opposite is true (see figure).



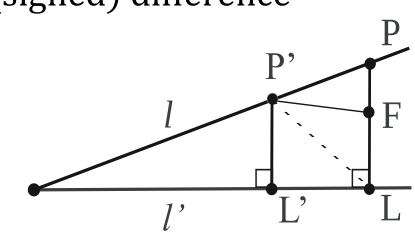
Proof. Let X be a point inside a parabola with focus F and directrix l, and let Y be the projection of point X on the directrix, i.e. a foot of the perpendicular to l from X, and this perpendicular intersects parabola at a point Z (see figure). Using the definition of a parabola, $|FZ| = |ZY|$, and the triangle inequality, $|FZ| > |FX| - |XZ|$ we obtain, $|FZ| = |ZY| > |FX| - |XZ|$, or, $|XY| > |FX|$. Similarly, if X is outside a parabola, $|FZ| < |FX| + |XZ|$, and, $|FZ| = |ZY| < |FX| + |XZ|$, or, $|XY| = |ZY| - |XZ| < |FX|$.

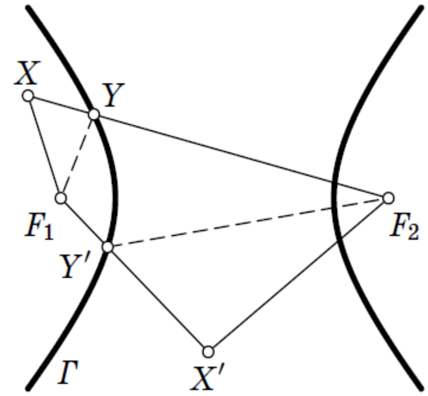
The optical property of the parabola. If a point light source, such as a small light bulb, is placed in the focus of a parabolic mirror, the reflected light forms plane-parallel beam perpendicular to the directrix (this is the principle used in spotlights). In other words, a light ray passing through one focus of a parabolic mirror upon reflection in such mirror will be perpendicular to the directrix of the parabola.



Proof. Suppose a line l is tangent to a parabola at a point P. Let P' be the projection of P to the directrix. Then, l is the bisector of the angle FPP' (see figure). Indeed, let point P belong to a parabola and l' be a bisector of the angle FPP', where $|PP'|$ is the distance to the directrix l. Then, for any point Q on l', $|FQ| = |QP'| \geq |QQ'|$. Hence, all points Q on l, except for $Q = P$, are outside the parabola, so l' is tangent to the parabola at point P.

Exercise. Consider the following problem. Given two lines, l and l', and a point F not on any of those lines, find point P on l such that the (signed) difference of distances from it to l' and F, $|P'L'| - |P'F|$, is maximal. As seen in the figure, for any P' on l the distance to l', $|P'L'| \leq |P'L| \leq |P'F| + |FL|$, where $|FL|$ is the distance from F to l'. Hence, $|P'L'| - |P'F| \leq |FL|$, and the difference is largest ($= |FL|$) when point P belongs to the perpendicular FL from point F to l'.

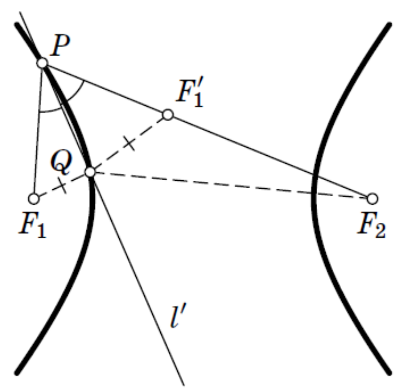




<u>The interior and exterior points of a hyperbola</u>. Let d be the difference of the distances from any point on the hyperbola to the foci F_1 and F_2 and let Γ be the branch of the hyperbola inside which F_1 lies. Then, for any point X inside (outside) Γ, the quantity $|XF_2| - |XF_1|$ is greater (less) than d (see figure).

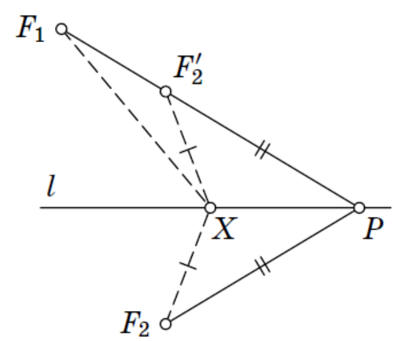
<u>Proof</u>. Let X be a point inside the branch Γ be of the hyperbola with the foci F_1 and F_2 and let Y be the intersection of the line XF_2 with the branch Γ. Using the definition of a hyperbola, $|YF_2| - |YF_1| = d$, and the triangle inequality, $|XF_1| < |XY| + |YF_1|$ we obtain, $|XF_2| - |XF_1| > |XF_2| - |XY| - |YF_1| = |YF_2| - |YF_1| = d$, or, $|XF_2| - |XF_1| > |YF_2| - |YF_1| = d$. Similarly, if X is outside the branch Γ of a hyperbola, $|XF_1| > |YF_1| - |XY|$, so $|XF_2| - |XF_1| < |YF_2| - |YF_1| = d$.

<u>The optical property of the hyperbola</u>. Suppose a line l is tangent to a hyperbola at a point P; then l is the bisector of the angle F_1PF_2, where F_1 and F_2 are the foci of the hyperbola (see figure). In other words, light ray passing through a focus, F_1, of a parabolic mirror upon reflection in such mirror will pass along the line that contains the other focus, F_2.

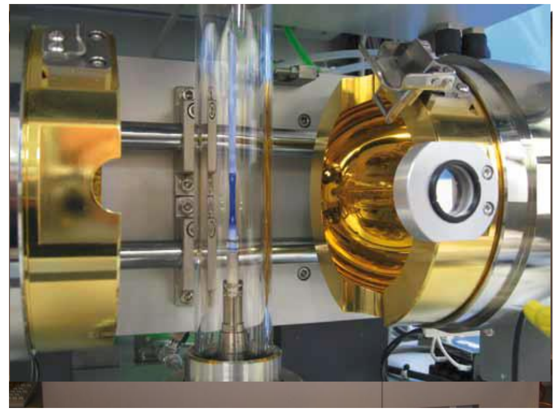


<u>Proof</u>. Let point P belong to a hyperbola with the foci F_1 and F_2, and line l' be a bisector of the angle F_1PF_2. Let $F_1{}'$ be the reflection of F_1 in l'. Then, for any point Q on l', $|QF_1| = |QF_1{}'|$, and $|QF_2| - |QF_1| = |QF_2| - |QF_1'| \le |F_2F_1'| = |PF_2| - |PF_1| = d$, again by the triangle inequality. Hence, all points Q on l', except for $Q = P$, are in-between the branches of the hyperbola, so l' is tangent to the hyperbola at point P.

<u>Exercise</u>. Consider the following problem. Given line l and points F_1 and F_2 lying on different sides of it, find point P on the line l such that the absolute value of the difference in distances from P to points F_1 and F_2 is maximal. As above, let $F_2{}'$ be the reflection of F_2 in l. Then for any point X on l, $|XF_2| - |XF_1'| \le |F_1F_2'|$.



Curves of the second degree around us.



If a light source is placed at one focus of an elliptic mirror, all light rays on the plane of the ellipse are reflected to the second focus. Since no other smooth curve has such a property, it can be used as an alternative definition of an ellipse. (In the special case of a circle with a source at its center all light would be reflected back to the center.) If the ellipse is rotated along its major axis to produce an ellipsoidal mirror (specifically, a prolate spheroid), this property will hold for all rays out of the source. Alternatively, a cylindrical mirror with elliptical cross-section can be used to focus light from a linear fluorescent lamp along a line of the paper; such mirrors are used in some document scanners. 3D elliptical mirrors are used in the floating zone furnaces to obtain locally high temperature needed for melting of the material for the crystal growth.

Sound waves are reflected in a similar way, so in a large elliptical room a person standing at one focus can hear a person standing at the other focus remarkably well.

In the 17th century, Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one focus, in his first law of planetary motion.

Giant hyperbolic mirrors in the Hubble Telescope.