

Homework for February 7, 2021.

### Algebra.

Review the last algebra classwork handouts. Solve the unsolved problems from the previous homeworks. Try solving the following problems.

1. Assume that the set of rational numbers  $\mathbb{Q}$  is divided into two subsets,  $\mathbb{Q}_<$  and  $\mathbb{Q}_>$ , such that all elements of  $\mathbb{Q}_>$  are larger than any element of  $\mathbb{Q}_<$ :  $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$ .
  - a. Prove that if  $\mathbb{Q}_>$  contains the smallest element,  $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 \leq b$ , then  $\mathbb{Q}_<$  does not contain the largest element
  - b. Prove that if  $\mathbb{Q}_<$  contains the largest element,  $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a \leq a_0$ , then  $\mathbb{Q}_>$  does not contain the smallest element
  - c. Present an example of such a partition, where neither  $\mathbb{Q}_>$  contains the smallest element, nor  $\mathbb{Q}_<$  contains the largest element
2. Prove the following properties of countable sets. For any two countable sets,  $A, B$ ,
  - a. Union,  $A \cup B$ , is also countable,  $((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)) \Rightarrow (c(A \cup B) = \aleph_0)$
  - b. Product,  $A \times B = \{(a, b), a \in A, b \in B\}$ , is also countable,  $((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)) \Rightarrow (c(A \times B) = \aleph_0)$
  - c. For a collection of countable sets,  $\{A_n\}, c(A_n) = \aleph_0$ , the union is also countable,  $c(A_1 \cup A_2 \dots \cup A_n) = \aleph_0$
3. Let  $W$  be the set of all “words” that can be written using the alphabet consisting of 26 lowercase English letters; by a “word”, we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that  $W$  is countable. [Hint: for any  $n$ , there are only finitely many words of length  $n$ .]
4. Compare the following real numbers (are they equal? which is larger?)
  - a.  $1.33333\dots = 1.(3)$  and  $4/3$
  - b.  $0.09999\dots = 0.0(9)$  and  $1/10$
  - c.  $99.9999\dots = 99.(9)$  and  $100$
  - d.  $\sqrt[2]{2}$  and  $\sqrt[3]{3}$
5. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).

- a.  $1/8$
  - b.  $2/7$
  - c.  $0.1$
  - d.  $0.33333... = 0.(3)$
  - e.  $0.13333... = 0.1(3)$
6. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

### Ordering and comparison.

1.  $\forall a, b \in \mathbb{R}$ , one and only one of the following relations holds
  - $a = b$
  - $a < b$
  - $a > b$
2.  $\forall a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$ , i.e.  $a < c < b$
3. Transitivity.  $\forall a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property.  $\forall a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$ , such that  $a < nb$

### Addition and subtraction.

- $\forall a, b \in \mathbb{R}, a + b = b + a$
- $\forall a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
- $\forall a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

## Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework.

### Problems.

1. Given two lines,  $l$  and  $l'$ , and a point  $F$  not on any of those lines, find point  $P$  on  $l$  such that the (signed) difference of distances from it to  $l'$  and  $F$ ,  $|P'L'| - |P'F|$ , is maximal. As seen in the figure, for any  $P'$  on  $l$  the distance to  $l'$ ,  $|P'L'| \leq |P'L| \leq |P'F| + |FL|$ , where  $|FL|$  is the distance from  $F$  to  $l'$ . Hence,  $|P'L'| - |P'F| \leq |FL|$ , and the difference is largest ( $= |FL|$ ) when point  $P$  belongs to the perpendicular  $FL$  from point  $F$  to  $l'$ .
2. Given line  $l$  and points  $F_1$  and  $F_2$  lying on different sides of it, find point  $P$  on the line  $l$  such that the absolute value of the difference in distances from  $P$  to points  $F_1$  and  $F_2$  is maximal. As above, let  $F_2'$  be the reflection of  $F_2$  in  $l$ . Then for any point  $X$  on  $l$ ,  $|XF_2| - |XF_1| \leq |F_1F_2'|$ .
3. Find the  $(x, y)$  coordinates of the common (intersection) point of the two lines, one passing through the origin at 45 degrees to the  $X$ -axis, and the other passing through the point  $(1,0)$  at 60 degrees to it.
4. Find the  $(x, y)$  coordinates of the common (intersection) points of the parabola  $y = x^2$  and of the ellipse centered at the origin and with major axis along the  $Y$ -axis whose length equals 2, and the minor axis along the  $X$ -axis whose length equals 1.
5. (Skanavi 10.122) Find the locus of the midpoints of all chords of a given circle with the center  $O$ , which intersect given chord  $AB$  of this circle.
6. Three circles of radius  $r$  touch each other. Find the area of the triangle  $ABC$  formed by tangents to pairs of circles (see figure).

