January 31, 2021 Math 9

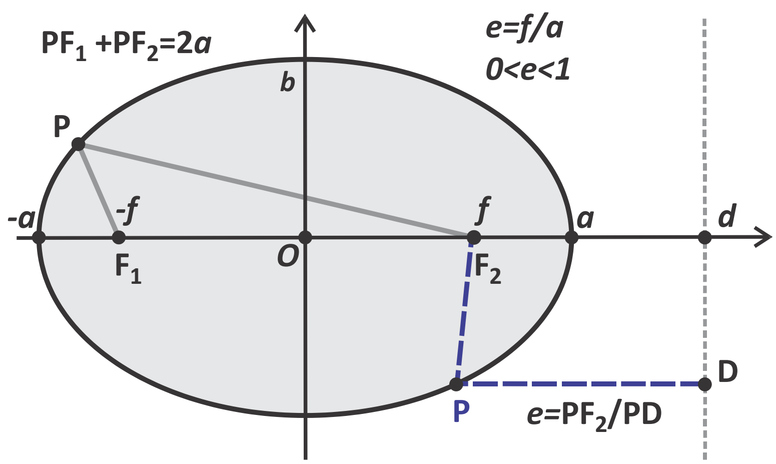
Geometry.

The method of coordinates (continued). Ellipse. Hyperbola. Parabola.

Ellipse.

Definition. An ellipse is a locus of points P, such that the sum of the distances from point P on the ellipse to the two given points, F_1 and F_2, is constant, $PF_1 + PF_2 = 2a$. Each of these two points is called the **focus** of the ellipse.

An ellipse is a smooth closed curve, which is symmetric about its horizontal and vertical axes. The distance between the antipodal points on the ellipse, or pairs of points whose midpoint is at the center of the ellipse, is maximum along the **major axis** or **transverse diameter**, and a minimum along the perpendicular **minor axis** or **conjugate diameter**. The **semi-major axis** (denoted by a in the figure) and the **semi-minor axis** (denoted by b in the figure) are one half of the major and minor axes, respectively. These are sometimes called (especially in technical fields) the **major** and **minor semi-axes**, or **major radius** and **minor radius**. The foci of the ellipse are two special points, F_1 and F_2 on the ellipse's major axis and are equidistant from the center point. The sum of the distances from point P on the ellipse to the foci is constant and equal to the major axis ($PF_1 + PF_2 = 2a$).



The **eccentricity** of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci, to the length of the major axis or $e = 2f/2a = f/a$. For an ellipse, the eccentricity is between 0 and 1 ($0 < e < 1$). When the eccentricity is 0 the foci coincide with the center point and the figure is a circle. As the eccentricity tends toward 1, the ellipse gets a more elongated shape. It tends towards a line segment if the two foci remain a finite distance apart and a parabola if one focus is kept fixed as the other is allowed to move arbitrarily far away. The distance $f = ae$ from a focal point to the centre is called the **linear eccentricity** of the ellipse.

Consider the locus of points such that the sum of the distances to two given points $F_1(-f, 0)$ and $F_2(f, 0)$ is the same for all points $P(x, y)$. It is an ellipse with the foci F_1 and F_2. This distance is equal to the length of the major axis of the ellipse, $2a$. Then, for every point on the ellipse,

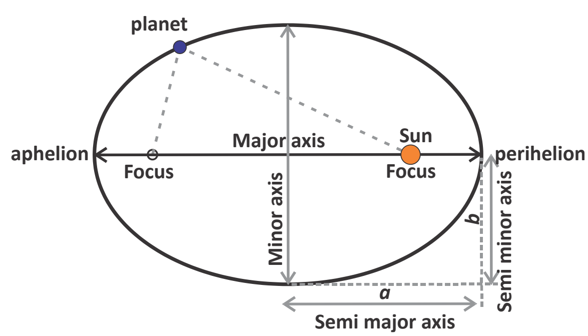
$$\sqrt{(x+f)^2 + y^2} + \sqrt{(x-f)^2 + y^2} = 2a.$$

By squaring this equation twice, we then obtain the equation of an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - f^2} = 1.$$

With the notation, $a^2 - f^2 = b^2$, the equation of an ellipse whose major and minor axes coincide with the Cartesian coordinate axes is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



The area enclosed by an ellipse is πab, where (as before) a and b are one-half of the ellipse's major and minor axes respectively. If the ellipse is given by the implicit equation, $x^2 + Bxy + Cy^2 = 1$, then the area is, $S = \frac{2\pi}{\sqrt{AC - B^2}}$.

Planets orbiting around the Sun follow elliptical trajectories.

Exercise. For the Earth's orbit, which is nearly circular, the semi major axis is 149.6 million km, the Perihelion is 147.09 million km and the Aphelion 152.1 million km. What is the semi minor axis of the Earth's orbit? What is its eccentricity?

In **stereometry**, an ellipse is defined as a plane curve that results from the intersection of a cone by a plane in a way that produces a closed curve. Circles are special cases of ellipses, obtained when the cutting plane is orthogonal to the cone's axis.

Hyperbola.

Definition. Hyperbola can be defined as the locus of points where the absolute value of the difference of the distances to the two foci is a constant equal to $2a$, the distance between its two vertices. This definition accounts for many of the hyperbola's applications, such as trilateration; this is the problem of determining position from the difference in arrival times of synchronized signals, as in GPS.

Similarly to the case of an ellipse, we write,

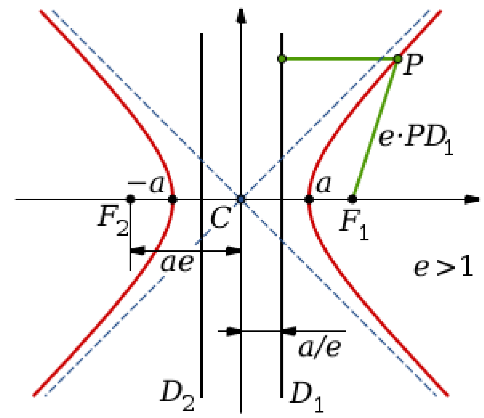
$$\sqrt{(x+f)^2+y^2}-\sqrt{(x-f)^2+y^2}=2a,$$

where $(f,0)$ and $(-f,0)$ are the positions of the two foci, which we chose to lie on the X-axis. Squaring the above equation twice, we arrive at the canonical equation for the hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{f^2-a^2}=1, \text{ or,}$$

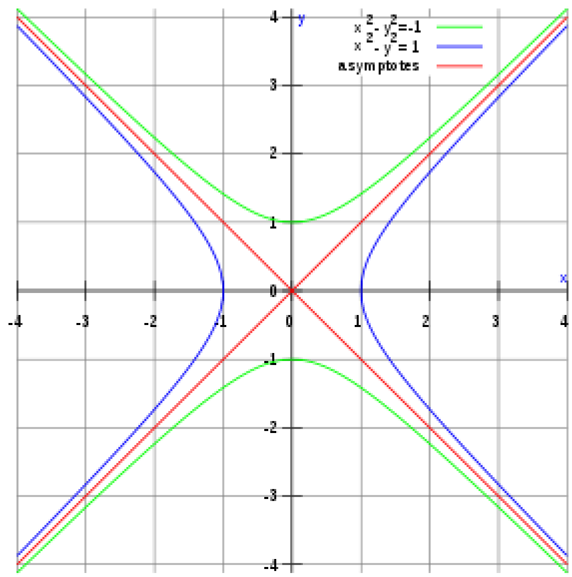
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ where } b^2=f^2-a^2\geq 0.$$

A hyperbola consists of two disconnected curves called its **arms**, or **branches**. At large distances from the center, the hyperbola approaches two lines, its asymptotes, which intersect at the hyperbola's center. A hyperbola approaches its asymptotes arbitrarily closely as the distance from its center increases, but it never intersects them; however, a degenerate hyperbola consists only of its asymptotes. Consistent with the symmetry of the hyperbola, if the transverse axis is aligned with the X-axis of a Cartesian coordinate system, the slopes of the asymptotes are equal in magnitude but



The asymptotes of the hyperbola (red curves) are shown as blue dashed lines and intersect at the center of the hyperbola, C. The two focal points are labeled F_1 and F_2, and the thin black line joining them is the transverse axis. The perpendicular thin black line through the center is the conjugate axis. The two thick black lines parallel to the conjugate axis (thus, perpendicular to the transverse axis) are the two directrices, D_1 and D_2. The eccentricity e equals the ratio of the distances from a point P on the hyperbola to one focus and its corresponding directrix line (shown in green). The two vertices are located on the transverse axis at $\pm a$ relative to the center. So the parameters are: a — distance from center C to either vertex; b — length of a perpendicular segment from each vertex to the asymptotes; c — distance from center C to either Focus point, F_1 and F_2, and θ — angle formed by each asymptote with the transverse axis.

opposite in sign, $\pm b/a$, where $b = a \times tan(\theta)$ and where θ is the angle between the transverse axis and either asymptote. The distance b (not shown) is the length of the perpendicular segment from either vertex to the asymptotes.



A **conjugate axis** of length $2b$, which in the case of an ellipse corresponds to the minor axis is sometimes drawn on the non-transverse principal axis; its endpoints $\pm b$ lie on the minor axis at the height of the asymptotes over/under the hyperbola's vertices. Because of the minus sign in some of the formulas below, it is also called the **imaginary axis** of the hyperbola.

If $b = a$, the angle 2θ between the asymptotes equals 90° and the hyperbola is said to be rectangular or equilateral. In this special case, the rectangle joining the four points on the asymptotes directly above and below the vertices is a square, since the lengths of its sides $2a = 2b$.

If the transverse axis of a hyperbola is aligned with the X-axis of a Cartesian coordinate system and is centered on the origin, the equation of the hyperbola can be written as,

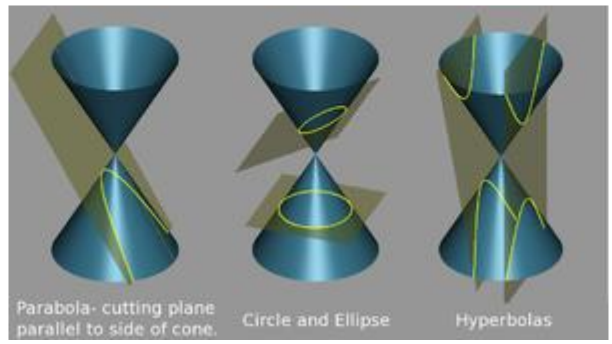
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

A hyperbola aligned in this way can be called an "East-West opening hyperbola". Likewise, a hyperbola with its transverse axis aligned with the y-axis can be called a "North-South opening hyperbola" and has equation,

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

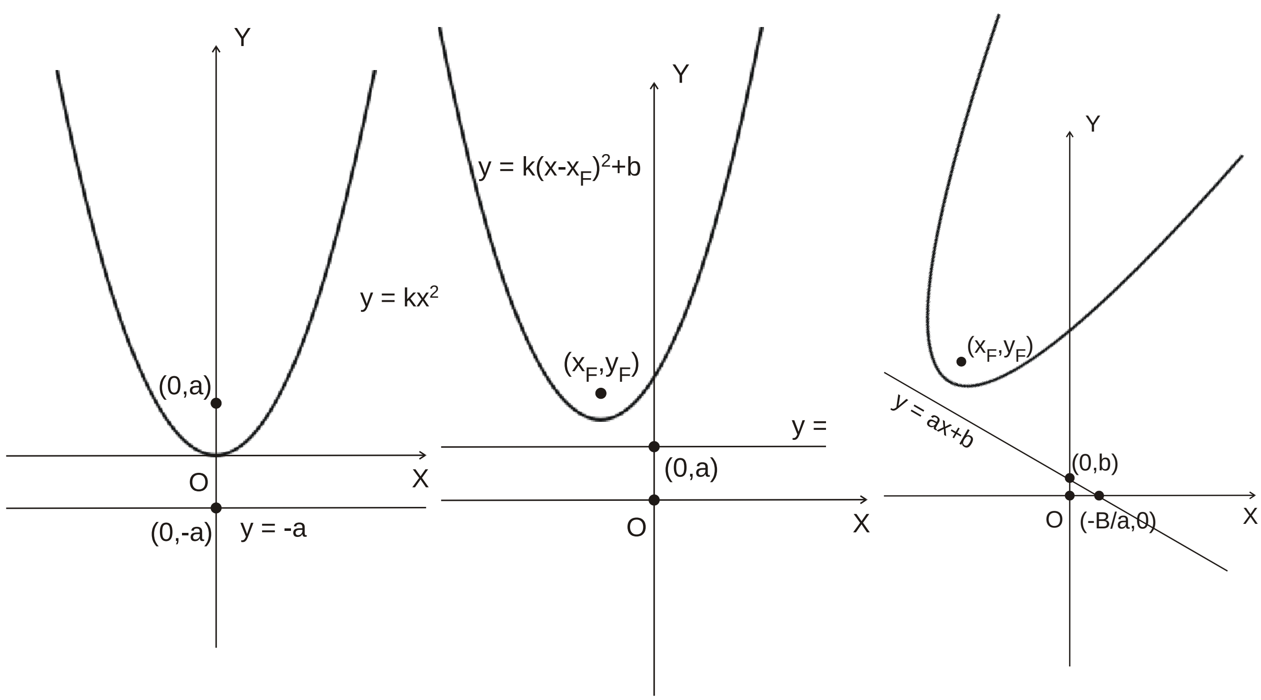
Every hyperbola is congruent to the origin-centered East-West opening hyperbola sharing the same eccentricity ε (its shape, or degree of "spread"), and is also congruent to the origin-centered North-South opening hyperbola with identical eccentricity ε — that is, it can be rotated so that it opens in the desired direction and can be translated (rigidly moved in the plane) so that it is centered at the origin. For convenience, hyperbolas are usually analyzed in terms of their centered East-West opening form.

In **stereometry**, hyperbola can be defined as a conic cross-section, similar to parabola and ellipse. Namely, hyperbola is the curve of intersection between a right circular conical surface and a plane that cuts through both halves of the cone. For the other major types of conic sections, the ellipse and the parabola, the plane cuts through only one half of the double cone. If the plane is parallel to the axis of the double cone and passes through its central apex, a degenerate hyperbola results that is simply two straight lines that cross at the apex point.



Parabola.

Definition. Parabola is A locus of points equidistant from a given point, F, and a given line, l, which does not contain this point. It is a parabola with focus F and the directrix l. The line perpendicular to the directrix and passing through the focus is its axis of symmetry; it is the line that splits the parabola through the middle. The point on the axis of symmetry that intersects the parabola is called the "vertex", and it is the point where the curvature is greatest.



The easiest way to show that the above definition indeed corresponds to a parabola is by using the method of coordinates. Indeed, let line l be parallel to the X-axis and intersect the Y-axis at $y = -a$, and focus $F(0, a)$ lie on the Y-axis at the same distance a from the origin. Then for any point on the parabola according to the definition above,

$x^2 + (y - a)^2 = (y + a)^2$, where from $y = \frac{1}{4a}x^2$.

If the directrix is a line $y = a$, and the focus has coordinates, $F(x_F, y_F)$, then the points on the parabola satisfy the equation $y = k(x - x_F)^2 + b$, where $k = \frac{1}{2(y_F - a)}$ and $b = \frac{y_F + a}{2}$. If the directrix is parallel to the Y-axis, then parabola's equation becomes $x = k\,(y - y_F)^2 + b$.

Parabolas can open up, down, left, right, or in some other arbitrary direction. Any parabola can be repositioned and rescaled to fit exactly on any other parabola — that is, all parabolas are similar. How do you think the equation of a parabola with the directrix $y = ax + b$, at an arbitrary angle $atan(a)$ to the (X, Y) axes, looks like? In order to solve this problem, we need to learn how to find a distance from the point $P(x, y)$ to a line $y = ax + b$.

In **stereometry**, parabola is defined as a conic section, similar to the ellipse and hyperbola. Parabola is a unique conic section, created from the intersection of a right circular conical surface and a plane parallel to a generating straight line of that surface. The parabola has many important applications, from automobile headlight reflectors to the design of ballistic missiles. They are frequently encountered in physics, engineering, and many other areas.