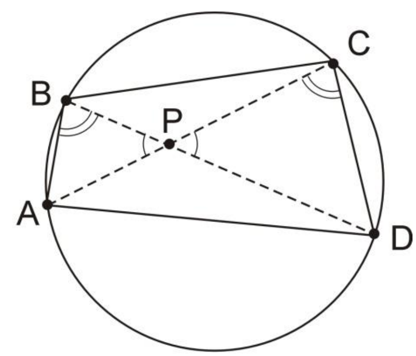
December 20, 2020

Geometry.

Recap: Corollaries of the Inscribed Angle Theorem. Euclids' theorems. Power of a point to a circle.

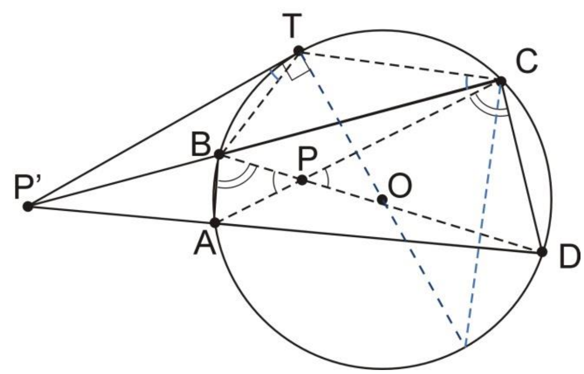
Consider the following figures. Using the theorem on the angle inscribed into a circle and the similarity of the corresponding triangles, it is easy to prove the following Euclid theorems.



1. If two chords, AC and BD intersect at a point P inside the circle, then

$|AP||PC| = |BP||PD| = R^2 - d^2,$

where R is the radius of the circle and and d is the distance from point P to the center of the circle, $d = |PO|$.



Proof. $\Delta APB \sim \Delta DPC$, so $\frac{|AP|}{|BP|} = \frac{|PD|}{|PC|}$, or,
$|AP||PC| = |BP||PD| = R^2 - d^2$.

2. If two chords, AD and BC intersect at a point P' outside the circle, then

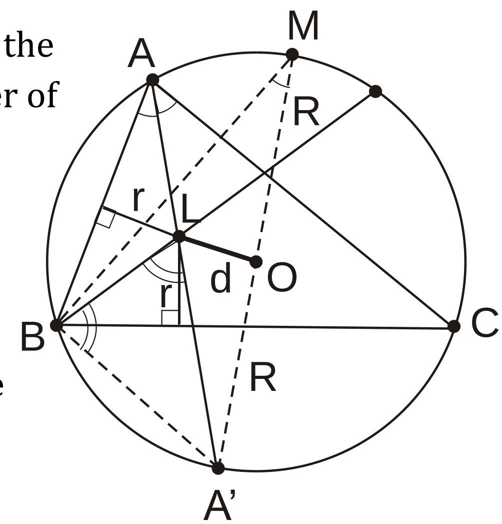
$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$

where $|PT|$ is a segment tangent to the circle.

Proof. $\Delta P'BD \sim \Delta P'AC$, so $\frac{|P'A|}{|P'B|} = \frac{|P'D|}{|P'C|}$, or, $|P'A||P'D| = |P'B||P'C|$.

For any circle of radius R and any point P distant d from the center, the quantity $d^2 - R^2$ is called the power of P with respect to the circle.

Application of the Euclids' theorems: Eulers' formula.



Using the above theorem the following formula for the distance between the incenter and the circumcenter of a triangle can be established.

Theorem. Let O and L be the circumcenter and the incenter (that is, the centers of the circumscribed and the inscribed circle, respectively) of a triangle ABC, with circumradius R and inradius r. Then, the distance $|OL| = d$ is given by

$$d^2 = R^2 - 2Rr.$$

Proof. Indeed, consider the figure, where the chord AA' passes through the incenter L, and the chord $A'M$ is the diameter of the circumcircle, passing through its center O. Triangle $A'MB$ is the right triangle by the inscribed angle theorem, and by the same theorem $\angle BAA' = \angle BMA'$. Hence, $\Delta A'BM$ is similar to the triangle with the hypotenuse AL whose leg is the radius of the inscribed circle (cf. Figure), so

$$|A'M| : |A'B| = |AL| : r.$$

Note that triangle $BA'L$ is isosceles, and therefore $|A'B| = |A'L|$. This is because $\angle A'LB = \angle ABL + \angle BAL$ as an external angle of ΔABL, while $\angle A'BL = \angle A'BC + \angle CBL = \angle A'AC + \angle CBL$ by the inscribed angle theorem, and $\angle BAL = \angle A'AC$ and $\angle ABL = \angle CBL$ since AL and BL are bisectors of $\angle BAC$ and $\angle CBA$, respectively (because L is the incenter).

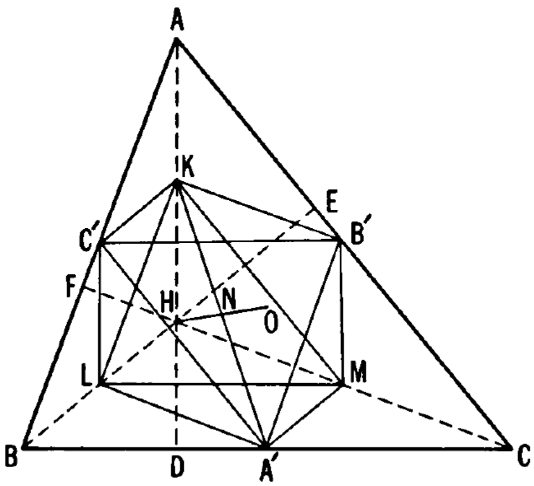
Substituting $|A'B| = |A'L|$ and $|A'M| = 2R$ in the above and using the Euclid theorem, $|AL||A'L| = R^2 - d^2$, we obtain,

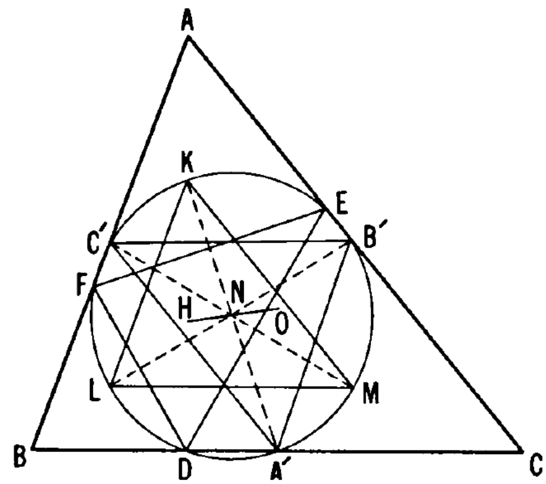
$$|AL||A'B| = |AL||A'L| = R^2 - d^2 = 2Rr,$$

which proves the above Euler's formula.

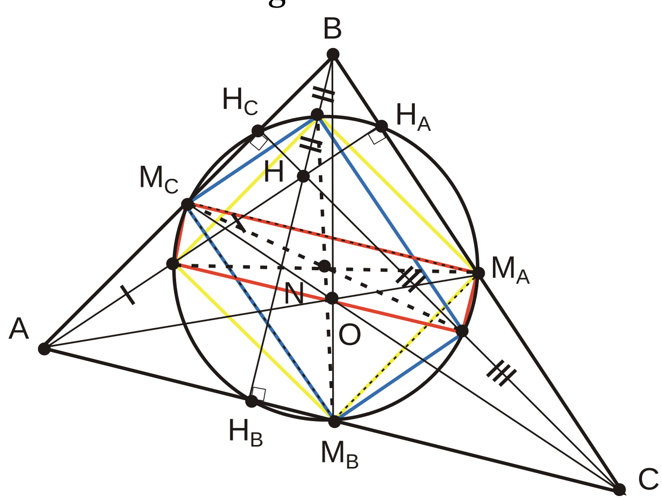
The nine-points circle problem.

Theorem. The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices lo the orthocenter, all lie on the same circle, of radius ½R.



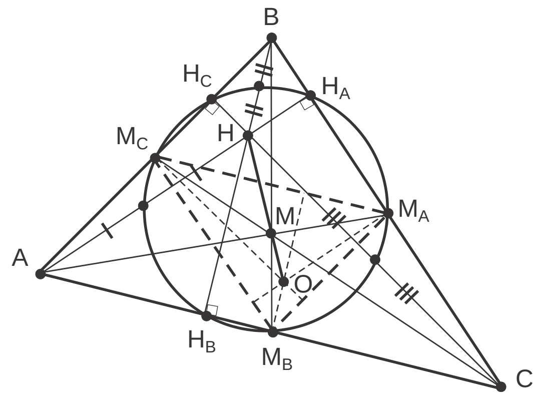


This theorem is usually credited to a German geometer Karl Wilhelm von Feuerbach, who actually rediscovered the theorem. The first complete proof appears to be that of Jean-Victor Poncelet, published in 1821, and Charles Brianson also claimed proving the same theorem prior to Feuerbach. The theorem also sometimes mistakenly attributed to Euler, who proved, as early as 1765, that the orthic triangle and the medial triangle have the same circumcircle, which is why this circle is sometimes called "the Euler circle". Feuerbach rediscovered Euler's partial result even later, and added a further property which is so remarkable that it has induced many authors to call the nine-point circle "the Feuerbach circle".



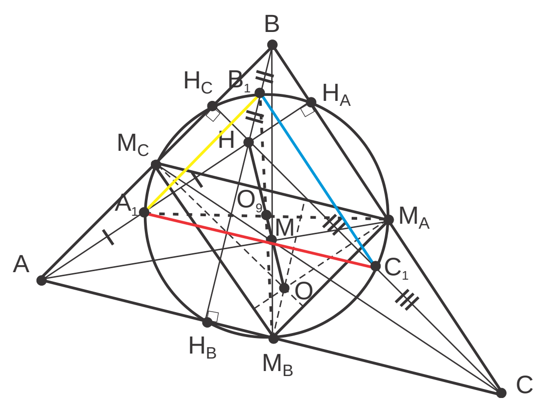
Proof. Consider rectangles formed by the mid-lines of triangle ABC and of triangles ABH, BCH and ACH.

Theorem. The orthocenter, H, centroid, M, and the circumcenter, O, of any triangle are collinear: all these three points lie on the same line, OH, which is called the Euler line of the triangle. The orthocenter divides the distance from the centroid to the circumcenter in 2: 1 ratio.



Proof. Note that the altitudes of the medial triangle$M_A M_B M_C$ are the perpendicular bisectors of the triangle ABC, so the orthocenter of $\Delta\ M_A M_B M_C$ is the circumcenter, O, of $\Delta\ ABC$. Now, using the property that centroid divides medians of a triangle in a 2:1 ratio, we note that triangles BMH and $M_B MO$ are similar, and homothetic with respect to point M, with the homothety coefficient 2.

Theorem. The center of the nine-point-circle lies on the (Euler's) line passing through orthocenter, centroid, and circumcenter, midway between the orthocenter and the circumcenter.



Proof. Consider the figure. Note the colored triangle $A_1 B_1 C_1$, which is formed by medians of triangles ABH, BHC and CHA, and is therefore congruent to the medial triangle $M_A M_B M_C$, but rotated 180 degrees. The 9 points circle is the circumcircle for both triangles, which means that rotation by 180 degrees about the center O_9 of the 9 point circle moves $\Delta\ M_A M_B M_C$ onto $\Delta\ A_1 B_1 C_1$, and the orthocenter, O, of the $\Delta\ M_A M_B M_C$ onto the orthocenter, H, of the $\Delta\ A_1 B_1 C_1$.