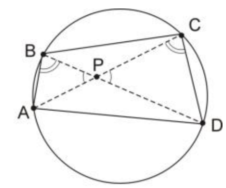
Homework for December 20, 2020.

Geometry.

Review the previous classwork notes. Solve the following problems, including problems from the last homework (if you have not solved them yet).

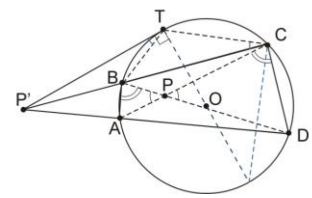
Problems.



1. Write the proof of the Euclid theorem, which states the following. If two chords AD and BC intersect at a point P' outside the circle, then

   $|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$

   where $|PT|$ is a segment tangent to the circle (see Figure).

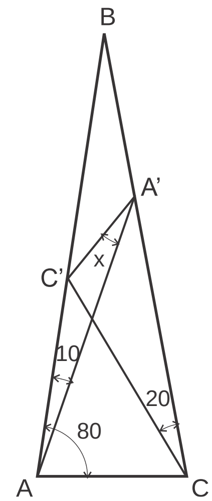


2. Using the Ptolemy's theorem, prove the following:
   a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
   b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.
3. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
4. Prove the Viviani's theorem:

   The sum of distances of a point P inside an equilateral triangle or on one of its sides, from the sides, equals the length of its altitude. Or, alternately,

   From a point P inside (or on a side) of an equilateral triangle ABC drop perpendiculars PP_a, PP_b, PP_c to its sides. The sum $|PP_a| + |PP_b| + |PP_c|$ is independent of P and is equal to any of the triangle's altitudes.

5. *In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



6. *Three Points are taken at random on an infinite plane. Find the chance of their being the vertices of an obtuse-angled Triangle. Hint: use the Viviani's theorem.
7. ** Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).

Algebra.

Review the last classwork handout. Review and solve the classwork exercises which were not solved and unsolved problems from the previous homeworks. Solve the following problems (skip the ones that you have already solved).

1. Prove the following properties of the Cartesian product,
   a. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   c. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
2. Find the Cartesian product, $A \times B$, of the following sets,
   a. $A = \{a, b\}$, $B = \{\uparrow, \downarrow\}$
   b. $A = \{June, July, August\}$, $B = \{1,15\}$
   c. $A = \emptyset$, $B = \{1,2,3,4,5,6,7,8,9\}$
3. Describe the set of points determined by the Cartesian product, $A \times B$, of the following sets (illustrate schematically on a graph),
   a. $A = [0,1]$, $B = [0,1]$ (two segments from 0 to 1)
   b. $A = [-1,1]$, $B = (-\infty, \infty)$
   c. $A = (-\infty, 0]$, $B = [0, \infty)$
   d. $A = (-\infty, \infty)$, $B = (-\infty, \infty)$
   e. $A = [0,1)$, $B = \mathbb{Z}$ (set of all integers)
4. Propose 3 meaningful examples of a Cartesian product of two sets.
5. $n_A = |A|$ is the number of elements in a set A.

a. What is the number of elements in a set $A \times A$
b. What is the number of elements in a set $A \times (A \times A)$

6. Find the following sum.

$$\left(2+\frac{1}{2}\right)^2 + \left(4+\frac{1}{4}\right)^2 + \cdots + \left(2^n + \frac{1}{2^n}\right)^2$$

7. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?
8. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

9. Find the following sum,
   a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
   b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n-1) \cdot 3^n$
10. Numbers $a_1, a_2, \dots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

11. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{1}{3-\sqrt{3}} + \frac{1}{6} + \cdots$$

12. What is the maximum value of the expression, $(1+x)^{36} + (1-x)^{36}$ in the interval $|x| \leq 1$?
13. Find the coefficient multiplying x^9 after all parenthesis are expanded in the expression, $(1+x)^9 + (1+x)^{10} + \cdots + (1+x)^{19}$.