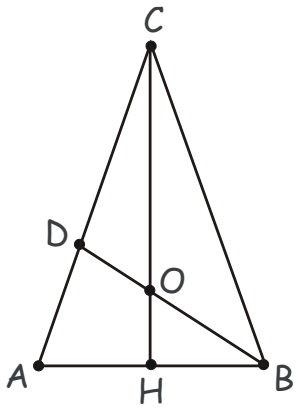
Homework for November 15, 2020.

Geometry.

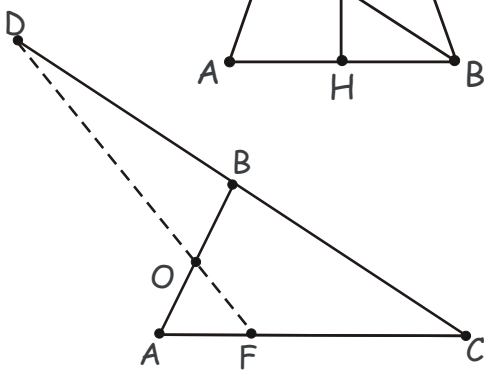
Review the classwork handout. Solve the unsolved problems from previous homeworks. Solve the following problems from the previous homework using the method of point masses and the Law of Lever (review the solutions for problems that were discussed in class).

Problems.

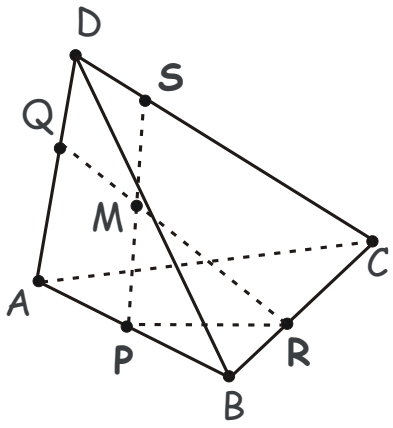
1. Prove that if a polygon has several axes of symmetry, they are all concurrent (cross at the same point).
2. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).
3. In isosceles triangle ABC point D divides the side AC into segments such that |AD|:|CD|=1:2. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio |OH| to |CH|.



4. Point D belongs to the continuation of side CB of the triangle ABC such that |BD| = |BC|. Point F belongs to side AC, and |FC| = 3|AF|. Segment DF intercepts side AB at point O. Find the ratio |AO|:|OB|.

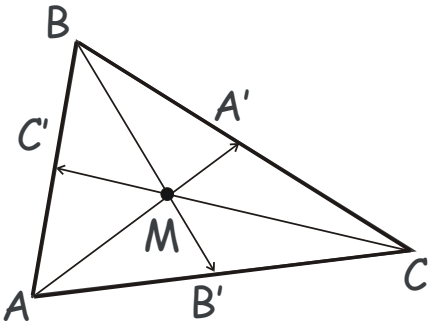


5. Consider segments connecting each vertex of the tetrahedron $ABCD$ with the centroid of the opposite face (the crossing point of its medians). Prove that all four of these segments, as well as the segments connecting the midpoints of the opposite edges (opposite edges have no common points; there are three pairs of opposite edges in a tetrahedron, and therefore three such segments) – seven segments in total, have common crossing point (are concurrent).



6. In a quadrilateral $ABCD$, E and F are the mid-points of its diagonals, while O is the point where the midlines (segments conneting the midpoints of the opposite sides) cross. Prove that E, F, and O are collinear (belong to the same line).

7. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



8. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?

9. In a parallelogram $ABCD$, a line passing through vertex D passes through a point E on the side AB, such that $|AE|$ is $1/n$-th of $|AB|$, n is an integer. At what distance from A, relative to the length, $|AC|$, of the diagonal AC it meets this diagonal?

Algebra.

Review the previous classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Using the method of mathematical induction, prove the following equality,

   $$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$$

2. Put the sign $<$, $>$, or $=$, in place of … below,

   $$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

3. Find the following sum.

   $$\left(2+\frac{1}{2}\right)^2 + \left(4+\frac{1}{4}\right)^2 + \cdots + \left(2^n+\frac{1}{2^n}\right)^2$$

4. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?
5. Solve the following equation,

   $\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

6. Find the following sum,
   a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
   b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n-1) \cdot 3^n$
7. Numbers $a_1, a_2, \ldots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

   $$S_n = a_1 a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

8. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{1}{3-\sqrt{3}}+\frac{1}{6}+\cdots$$

9. What is the maximum value of the expression, $(1+x)^{36}+(1-x)^{36}$ in the interval $|x| \leq 1$?
10. Find the coefficient multiplying x^9 after all parenthesis are expanded in the expression, $(1+x)^9+(1+x)^{10}+\cdots+(1+x)^{19}$.