Homework for October 11, 2020.

Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework. Solve the following problems (you may skip the ones considered in class).

1. Using logical equivalence, $\sim(A \Rightarrow B) \Leftrightarrow (A \wedge \sim(B))$, consider/prove,

$$\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$$

2. Let P, Q and R be some logical propositions. Write a negation of the logical expression and simplify the result as much as possible,
   a. $((P \wedge Q) \vee (R \wedge Q)$
   b. $((P \vee R) \wedge (Q \vee R))$
   c. $(P \Leftrightarrow (Q \wedge R))$
   d. $((P \wedge Q) \Rightarrow (Q \vee R))$
   e. $((P \vee Q) \Leftrightarrow (Q \vee R))$
3. Solve the following equations:
   a. $\frac{x-a}{x-b} + \frac{x-b}{x-a} = 2.5$
   b. $\sqrt{3x+4} + \sqrt{x-2} = 2\sqrt{x}$
   c. $\frac{1}{x^3+2} - \frac{1}{x^3+3} = \frac{1}{12}$
   d. $\frac{1}{x^2} + \frac{1}{(x+2)^2} = \frac{10}{9}$
   e. $\sqrt{x-2} = x - 4$
   f. $1 + \sqrt{1 + x\sqrt{x^2 - 24}} = x$
   g. $\sqrt{x} + \frac{2x+1}{x+2} = 2$
4. Simplify expressions:
   a. $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$
   b. $\frac{\sqrt{\sqrt{\frac{x-1}{x+1}} + \sqrt{\frac{x+1}{x-1}} - 2}}{\sqrt{(x+1)^3} - \sqrt{(x-1)^3}}\left(2x + \sqrt{x^2-1}\right)$
   c. $\frac{\sqrt{x - 4\sqrt{x-4}} + 2}{\sqrt{x + 4\sqrt{x-4}} - 2}$

5. Prove that:

a. $\frac{\sqrt{7+4\sqrt{3}}\cdot\sqrt{19-8\sqrt{3}}}{4-\sqrt{3}}-\sqrt{3}=2$

b. $\sqrt{6a+2\sqrt{9a^2-b^2}}-\sqrt{6a-2\sqrt{9a^2-b^2}}=2\sqrt{3a-b}$

6. **Recap.** There are 15 students in a class.

a. Each student needs to make a presentation of a problem. How many ways are there to arrange the order in which they do presentations – i. e. decide who speaks first, who speaks second, third, …, fifteenth?

b. How many ways is there to select a pair of students of whom one will present solution of an algebra problem, and the other of a problem in geometry?

c. How many ways is there to select a team of two students who will represent the class at a math competition?

7. **Recap.** Binomial coefficients are defined by

$$C_n^k = {}_kC_n = \binom{n}{k} = \frac{n!}{k!\,(\,n-k)!}$$

Prove that binomial coefficients satisfy the following identities,

$$C_n^0 = C_n^n \Leftrightarrow \binom{n}{0} = \binom{n}{n} = 1$$

$$C_n^k = C_n^{n-k} \Leftrightarrow \binom{n}{k} = \binom{n}{n-k}$$

$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k \Leftrightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$C_{n+1}^k = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$C_n^{k+1} = \binom{n}{k+1} = \binom{n}{k}\frac{n-k}{k+1}$$

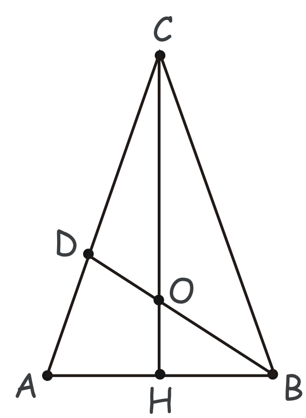
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Geometry.

Review the classwork handout. Solve the remaining problems from the previous homework. In particular, review constructing the segments equal to the ratio and the product of two given segments. Try solving the following additional problems. Although we will review the Thales theorem next time – you may assume that it had been already proven.

Problems.

1. Midsegment of a triangle is a line segment joining the midpoints of two sides. Prove that midsegment of a triangle is parallel to the third side and its length is 1/2 of that side's length.
2. Using a ruler and a compass, construct a triangle given the midpoints of its three sides.
3. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).
4. In isosceles triangle ABC point D divides the side AC into segments such that |AD|:|CD|=1:2. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio |OH| to |CH|.



5. In a trapezoid $ABCD$ with the bases $|AB| = a$ and $|CD| = b$, segment MNparallel to the bases, $MN||AB$, connects the opposing sides, $M \in [AD]$ and $N \in [BC]$. MN also passes through the intersection point O of the diagonals, AC and BD, as shown in the Figure. Prove that $|MN| = \frac{2ab}{a+b}$.