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Algebra. Basic Notations.

Elements of Mathematical Logic.

Proposition is a sentence that is either true, or false, but not both. For example,

"Grass is green", and "$2 + 5 = 5$" are propositions.

"Close the door", and "Is it hot outside?" are not propositions.

Also, "x is greater than 2", where x is a variable representing a number, is not a proposition, because unless a specific value is given to x we cannot say whether it is true or false.

The Propositional Logic

The propositional logic constitutes the foundation of the full logical formalism. It provides only the **connectives** operations ∧, ∨, ~, →, and ↔ and the two (propositional valuation) constants 'true' and 'false'.

Simple sentences, which are true, or false, are basic propositions. Larger and more complex sentences are constructed from basic propositions by combining them with connectives.

NOT negation	AND conjunction	OR disjunction	IF_THEN sufficient	ONLY_IF necessary	IF_AND_ONLY_IF equivalent
¬,~,$\bar{\ }$	∧	∨	→,⇒	←,⇐	↔, ⇔

In everyday life we often combine propositions to form more complex propositions without paying much attention to them.

Let **X** represents proposition "It is raining", and **Y** represents proposition "Joe takes his umbrella". Then [~ X] – negation, [X ∧Y] – conjunction, [X ∨Y] – disjunction, [X →Y] and [X ←Y] – conditional and [X ↔Y] – equivalence – are propositions.

Exercise. Consider the following "truth tables" for propositions obtained by applying logical operations and understand their meaning.

X	~X
T	F
F	T

X	Y	X∧Y
T	T	T
F	T	F
T	F	F
F	F	F

X	Y	X∨Y
T	T	T
F	T	T
T	F	T
F	F	F

X	Y	X↔Y
T	T	T
F	T	F
T	F	F
F	F	T

X	Y	X→Y
T	T	T
F	T	T
T	F	F
F	F	T

When $A \rightarrow B$ is always true, B follows from A, we express that by $A \Rightarrow B$.

When $A \leftarrow B$ is always true, A follows from B, we express that by $A \Leftarrow B$.

When $A \leftrightarrow B$ is always true, A and B are equivalent, we express that by $A \Leftrightarrow B$.

These two conditional claims, "If A, then B" and "A, only if B" refer to two different kinds of conditions: a **sufficient** condition and a **necessary** condition.

A **sufficient** condition is one that, if satisfied, assures the statement's truth. "If A, then B". If A is truth, then B is also truth, ***A* is sufficient for *B***. If we have A, then we know that B must follow, $A \Rightarrow B$.

Example. Earning a total of 950 points (95%) in English class is a sufficient condition for earning a final grade of A. If you have 950 points, then it follows that you will have a final grade of A. It is not necessary to earn 950 points to earn an A in the English class. You can earn 920 points to earn an A. (We cannot say that if you do not have 950 points then you can't have an A.)

A **necessary** condition of a statement must be satisfied for the statement to be true. "A, only if B" means ***B* is necessary for *A***, $B \Leftarrow A$. If we do not have B, then we will not have A.

Example. I need to put gasoline into my car, without it I will not be able to start the engine. Of course, having gasoline in the car does not guarantee that my car will start. There are many other conditions needed for my car to start, but if there is no gasoline it will definitely not going anywhere.

If A is **sufficient** for B, $A \Rightarrow B$, then B is **necessary** for A.

Is sunlight a necessary or sufficient condition for the roses to bloom?

Is earning a final grade of C a necessary or sufficient condition for passing the course?

Is being a male a necessary or sufficient condition for being a father?

Is attending class regularly and punctually a necessary or sufficient condition for being successful in class?

Is being 20 years old a necessary or sufficient condition for being a college student?

Is completing all the requirements of your degree program a necessary or sufficient condition for earning your degree?

Definition. A and B are equivalent if both conditions $A \Rightarrow B$ **and** $A \Leftarrow B$ take place. A is necessary **and** sufficient for B, and vice versa. Then we write $A \Leftrightarrow B$.

Definition. The contrapositive of the conditional statement has its antecedent and consequent inverted and flipped: the contrapositive of $A \Rightarrow B$ is $\sim B \Rightarrow \sim A$, or $\sim A \Leftarrow \sim B$. If negation of B is truth, then negation of A will be truth.

Contraposition is a law that says that a conditional statement is logically equivalent to its **contrapositive**, $(A \Rightarrow B) \Leftrightarrow (\sim B \Rightarrow \sim A)$.

Tautologies, Axioms and Inference Rules.

Definition. A proposition F is a **tautology** if it evaluates to truth for all the assignments covering it.

So tautologies are propositional formulae which possess 'universal logical validity' and evaluate to true no matter what truth values are assigned to their variables. Examples are

$p \vee (\sim p)$, $q \rightarrow (p \rightarrow q)$, $p \rightarrow (q \rightarrow (p \wedge q))$.

That a proposition is a tautology can be determined by evaluating its possible values, e. g. using the truth tables. Alternatively, a set of tautologies can be adopted as axioms and inference rules, from which all other propositions are derived. The axioms and inference rules can be chosen in many ways. Though not at all the smallest, the following is one possible set, having a familiar and convenient algebraic flavor.

(i) $(p \wedge q) \leftrightarrow (q \wedge p)$

(ii) $((p \wedge q) \wedge r) \leftrightarrow (p \wedge (q \wedge r))$

(iii) $(p \wedge p) \leftrightarrow p$

(iv) $(p \vee q) \leftrightarrow (q \vee p)$

(v) $((p \vee q) \vee r) \leftrightarrow (p \vee (q \vee r))$

(vi) $(p \vee p) \leftrightarrow p$

(vii) $(\sim(p \wedge q)) \leftrightarrow ((\sim p) \vee (\sim q))$ – **De Morgan's law 1**

(viii) $(\sim(p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$ – **De Morgan's law 2**

(ix) $((p \vee q) \wedge r) \leftrightarrow ((p \wedge r) \vee (q \wedge r))$

(x) $((p \wedge q) \vee r) \leftrightarrow ((p \vee r) \wedge (q \vee r))$

(xi) $(p \leftrightarrow q) \rightarrow ((p \wedge r) \leftrightarrow (q \wedge r))$

(xii) $(p \leftrightarrow q) \rightarrow ((p \vee r) \leftrightarrow (q \vee r))$

(xiii) $(p \leftrightarrow q) \rightarrow ((\sim p) \leftrightarrow (\sim q))$

(xiv) $(p \leftrightarrow q) \rightarrow (q \rightarrow p)$

(xv) $(p \rightarrow q) \leftrightarrow ((\sim p) \vee q)$

(xvi) $(p \leftrightarrow q) \leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p))$

(xvii) $(p \wedge q) \rightarrow p$

(xviii) $(p\leftrightarrow q)\rightarrow((q \leftrightarrow r)\rightarrow(p\leftrightarrow r))$

(xix) $(p\leftrightarrow q)\rightarrow(q \leftrightarrow p)$

(xx) $(p\leftrightarrow p)$

(xxi) $(p \wedge (\sim p))\leftrightarrow$false

(xxii) $(p \vee (\sim p))\leftrightarrow$true

(xxiii) $(\sim(\sim p))\leftrightarrow p$

(xxiv) $(p \wedge$ true$)\leftrightarrow p$

(xxv) $(p \wedge$ false$)\leftrightarrow$false

(xxvi) $(p \vee$ true$)\leftrightarrow$true

(xxvii) $(p \vee$ false$)\leftrightarrow p$

(xxviii) $(\sim$true$)\leftrightarrow$false

(xxix) $(\sim$false$)\leftrightarrow$true

(xxx) true

It is easy to verify that all these are in fact tautologies. This large set of axioms needs to be supplemented with only one rule of inference, the **modus ponens** of mediaeval logicians.

<u>**Modus ponens**</u>: From two formulae of the form p and $p \rightarrow q$, infer q.

Alternatively, a propositional calculus can be based upon a **set of inference rules**, from which all other statements are derived.

1. <u>Negation introduction</u>: $\{(p \rightarrow q), (p \rightarrow \sim q)\} \Rightarrow \sim p$. From $(p \rightarrow q)$ and $(p \rightarrow \sim q)$, infer $\sim p$.
2. <u>Negation elimination</u>: $\{\sim p\} \Rightarrow (p \rightarrow q)$. From $\sim p$, infer $(p \rightarrow q)$.
3. <u>Double negative elimination</u>: $\sim\sim p \Rightarrow p$. From $\sim\sim p$, infer p.
4. <u>Conjunction introduction</u>: $\{p, q\} \Rightarrow (p \wedge q)$. From p and q, infer $(p \wedge q)$.

5. Conjunction elimination: $(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$. From $(p \wedge q)$, infer p. From $(p \wedge q)$, infer q.
6. Disjunction introduction: $p \Rightarrow (p \vee q)$ and $q \Rightarrow (p \vee q)$. From p, infer $(p \vee q)$. From q, infer $(p \vee q)$.
7. Disjunction elimination: $\{(p \vee q), (p \rightarrow r), (q \rightarrow r)\} \Rightarrow r$. From $(p \vee q)$ and $(p \rightarrow r)$ and $(q \rightarrow r)$, infer r.
8. Biconditional introduction: $\{(p \rightarrow q), (q \rightarrow p)\} \Rightarrow (p \leftrightarrow q)$. From $(p \rightarrow q)$ and $(q \rightarrow p)$, infer $(p \leftrightarrow q)$.
9. Biconditional elimination: $(p \leftrightarrow q) \Rightarrow (p \rightarrow q)$ and $(p \leftrightarrow q) \Rightarrow (q \rightarrow p)$. From $(p \leftrightarrow q)$, infer $(p \rightarrow q)$. From $(p \leftrightarrow q)$, infer $(q \rightarrow p)$.
10. Modus ponens **(conditional elimination)**: $\{p, (p \rightarrow q)\} \Rightarrow q$. From p and $(p \rightarrow q)$, infer q.
11. Conditional proof **(conditional introduction)**: $(p \Rightarrow q) \Rightarrow (p \rightarrow q)$. From [accepting p allows a proof of q], infer $(p \rightarrow q)$.

A list of basic and derived argument forms and logical equivalences is given at the end.

Predicate Calculus. Quantifiers.

Predicate Calculus is the branch of formal logic, also called functional calculus, which deals with representing the logical connections between statements as well as the statements themselves. The predicate calculus enlarges the propositional logic (calculus), preserving all its operations, but also allowing compound functional and predicate terms and the two quantifiers, $\forall$ and $\exists$.

A **predicate** is a verb phrase that describes a property of objects, or a relationship among objects represented by the variables, and generalizes the concept "proposition". We can use notation $P(x, y)$, where P is a predicate name, and x and y denote objects or variables. Informally, a predicate is a statement that may be true or false depending on the values of its variables.

Definition. **Predicate** with variables is a **proposition** if,

- a value is assigned to the variable,

or

- possible values of the variable are quantified using a **quantifier**.

Example. For $x > 1$ to be a proposition, either we substitute a specific number for x, or change it to something like "There is a number x for which $x > 1$ holds", or, using a quantifier, $\exists x, x > 1$.

<u>Quantifiers</u>.

$\exists$ is called the **existential quantifier**, and reads " ... there exists ...".

$\exists x \in X$: ... $\Leftrightarrow$ "there exists an x in the set X such that ..."

For example, "someone loves you" could be transformed into the propositional form, $\exists x \in X: P(x)$, where:

- $P(x)$ is the predicate meaning: ***x* loves you**,
- Set of objects of interest X includes (not limited to) all living creatures.

The statement $D(x)$, "equation $x^3 + 3x^2 + 5x + 15 = 0$ has a real solution", can be written in a predicate form as, $\exists x \in R: x^3 + 3x^2 + 5x + 15 = 0$.

Exercise. Construct negation for $P(x)$ and $D(x)$.

$\forall$ is called the **universal quantifier**, and reads " for all".

$\forall x \in X$: ... $\Leftrightarrow$ " for all x in the set X ..."

Example 1. "All cars have wheels" could be transformed into the propositional form, $\forall\{x, (x\ is\ a\ car)\}: D(x)$, where,

- $D(x)$ is the predicate denoting: ***x* has wheels**, and
- Set of objects of interest, X, is only populated by cars.

Example 2. $\forall x, x < x^2$. Is this true or false? How we fix it if we should?

These two quantifiers (plus the usual logical operations such as conjunction and disjunction, i.e. AND, OR,...) are sufficient to write all statements in math.

Predicate Negation Laws. [Generalized De Morgan]

$$\sim (\exists x \in X: p_i) \equiv \forall x \in X: \sim p_i$$

$$\sim (\forall x \in X: p_i) \equiv \exists x \in X: \sim p_i$$

<u>**Logic and Proofs with Quantifiers**</u>

If you know that $\forall x \in X: P(x)$, where $P(x)$ is some statement, and $y \in X$, then you can conclude that $P(y)$ is true.

Example. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

Another common method of reasoning: if you have the statement $\exists\, x: P(x)$, you can say "let us choose x such that $P(x)$".

Example. Some students in this class are in Math Theory. All students in math theory are smart. Therefore, some students in this class are smart.

Indeed, we know that some students in this class are in Math Theory; let S be one of these students. Then, by the second statement, S is smart; so we have a student in our class who is smart.

Rules of logic involving quantifiers are called Syllogisms. The first person who tried to give a full list of such rules was Aristotle, but his system was extremely complicated. It had 24 syllogisms, each of which was given a proper name (Barbara, Darii, etc) and could be in several forms (figures). Here is an example of one of his syllogisms (Darii):

All rabbits have fur. Some pets are rabbits. Therefore, some pets have fur.

<u>**Proofs with quantifiers**</u>

To prove a statement of the form $\exists x \in X: P(x)$, it suffices to produce one example. It is not logically necessary to explain how you found this example (even though your teacher might ask you for this). For example, to prove $\exists x \in R: x^3 + 3x^2 + 5x + 15 = 0$, it suffices to say "take $x = -3$; then $(-3)^3 + 3(-3)^2 + 5(-3) + 15 = 0$".

To disprove such a statement, you need to explain why there is no x for which $P(x)$ holds. It is usually a hard task; sometimes you can do it by contradiction (assume that for some x, the statement $P(x)$ is true; then....).

To prove a statement of the form $\forall x \in X: P(x)$, you need to give an argument which works for all $x \in X$. **It is not enough to consider several examples!** (But it can be useful in trying to find a general argument). To disprove a statement of the form $\forall x \in : P(x)$, it suffices to produce one example when this statement is false. For example, to disprove the statement $\forall n \in N: P(x): n^2 + n + 41$ is prime (where N is the set of positive integers), it suffices to give one example: for $n = 41, n^2 + n + 41$ is a multiple of 41 and thus not prime.

<u>**A summary of logical equivalences**</u>.

Commutative laws:

1. $(A \wedge B) \Leftrightarrow (B \wedge A)$
2. $(A \vee B) \Leftrightarrow (B \vee A)$
3. $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow A)$

Associative laws:

1. $\big(A \wedge (B \wedge C)\big) \Leftrightarrow \big((A \wedge B) \wedge C\big)$
2. $\big(A \vee (B \vee C)\big) \Leftrightarrow \big((A \vee B) \vee C\big)$
3. $\big(A \Leftrightarrow (B \Leftrightarrow C)\big) \Leftrightarrow \big((A \Leftrightarrow B) \Leftrightarrow C\big)$

Distributive laws:

4. $\big(A \wedge (B \vee C)\big) \Leftrightarrow \big((A \wedge B) \vee (A \wedge C)\big)$
5. $\big(A \vee (B \wedge C)\big) \Leftrightarrow \big((A \vee B) \wedge (A \vee C)\big)$
6. $\big(A \Rightarrow (B \wedge C)\big) \Leftrightarrow \big((A \Rightarrow B) \wedge (A \Rightarrow C)\big)$
7. $\big(A \Rightarrow (B \vee C)\big) \Leftrightarrow \big((A \Rightarrow B) \vee (A \Rightarrow C)\big)$
8. $\big((A \wedge B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \vee (B \Rightarrow C)\big)$
9. $\big((A \vee B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \wedge (B \Rightarrow C)\big)$

Negation laws:

1. $\sim(A \wedge B) \Leftrightarrow \big(\sim(A) \vee \sim(B)\big)$
2. $\sim(A \vee B) \Leftrightarrow \big(\sim(A) \wedge \sim(B)\big)$
3. $\sim(\sim A) \Leftrightarrow A$
4. $\sim(A \Rightarrow B) \Leftrightarrow \big(A \wedge \sim(B)\big)$
5. $\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$
6. $\sim(A \Leftrightarrow B) \Leftrightarrow \big(A \Leftrightarrow \sim(B)\big)$

Implication laws:

1. $(A \Rightarrow B) \Leftrightarrow \left(\sim\left(A \wedge \sim(B)\right)\right)$
2. $(A \Rightarrow B) \Leftrightarrow (\sim(A) \vee B)$
3. $(A \Rightarrow B) \Leftrightarrow \left(\sim(B) \Rightarrow \sim(A)\right)$
4. $(A \Leftrightarrow B) \Leftrightarrow \left((A \Rightarrow B) \wedge (B \Rightarrow A)\right)$
5. $(A \Leftrightarrow B) \Leftrightarrow \left(\sim(A) \Leftrightarrow \sim(B)\right)$

Table of Basic and Derived Argument Forms (⊢ and ⇒ mean "infer", "entail").

Basic and Derived Argument Forms		
Name	**Sequent**	**Description**
Modus Ponens	$((p \to q) \wedge p) \vdash q$	If *p* then *q*; *p*; therefore *q*
Modus Tollens	$((p \to q) \wedge \neg q) \vdash \neg p$	If *p* then *q*; not *q*; therefore not *p*
Hypothetical Syllogism	$((p \to q) \wedge (q \to r)) \vdash (p \to r)$	If *p* then *q*; if *q* then *r*; therefore, if *p* then *r*
Disjunctive Syllogism	$((p \vee q) \wedge \neg p) \vdash q$	Either *p* or *q*, or both; not *p*; therefore, *q*
Constructive Dilemma	$((p \to q) \wedge (r \to s) \wedge (p \vee r)) \vdash (q \vee s)$	If *p* then *q*; and if *r* then *s*; but *p* or *r*; therefore *q* or *s*
Destructive	$((p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)) \vdash (\neg p \vee \neg r)$	If *p* then *q*;

Dilemma		and if *r* then *s*; but not *q* or not *s*; therefore not *p* or not *r*
Bidirectional Dilemma	$((p \to q) \land (r \to s) \land (p \lor \neg s)) \vdash (q \lor \neg r)$	If *p* then *q*; and if *r* then *s*; but *p* or not *s*; therefore *q* or not *r*
Simplification	$(p \land q) \vdash p$	*p* and *q* are true; therefore *p* is true
Conjunction	$p, q \vdash (p \land q)$	*p* and *q* are true separately; therefore they are true conjointly
Addition	$p \vdash (p \lor q)$	*p* is true; therefore the disjunction (*p* or *q*) is true
Composition	$((p \to q) \land (p \to r)) \vdash (p \to (q \land r))$	If *p* then *q*; and if *p* then *r*; therefore if *p* is true then *q* and *r* are true

De Morgan's Theorem (1)	$\neg(p \land q) \vdash (\neg p \lor \neg q)$	The negation of (*p* and *q*) is equiv. to (not *p* or not *q*)
De Morgan's Theorem (2)	$\neg(p \lor q) \vdash (\neg p \land \neg q)$	The negation of (*p* or *q*) is equiv. to (not *p* and not *q*)
Commutation (1)	$(p \lor q) \vdash (q \lor p)$	(*p* or *q*) is equiv. to (*q* or *p*)
Commutation (2)	$(p \land q) \vdash (q \land p)$	(*p* and *q*) is equiv. to (*q* and *p*)
Commutation (3)	$(p \leftrightarrow q) \vdash (q \leftrightarrow p)$	(*p* is equiv. to *q*) is equiv. to (*q* is equiv. to *p*)
Association (1)	$(p \lor (q \lor r)) \vdash ((p \lor q) \lor r)$	*p* or (*q* or *r*) is equiv. to (*p* or *q*) or *r*
Association (2)	$(p \land (q \land r)) \vdash ((p \land q) \land r)$	*p* and (*q* and *r*) is equiv. to (*p* and *q*) and *r*
Distribution (1)	$(p \land (q \lor r)) \vdash ((p \land q) \lor (p \land r))$	*p* and (*q* or *r*) is equiv. to (*p* and *q*) or (*p* and *r*)
Distribution	$(p \lor (q \land r)) \vdash ((p \lor q) \land (p \lor r))$	*p* or (*q* and

(2)		*r*) is equiv. to (*p* or *q*) and (*p* or *r*)
Double Negation	$p \vdash \neg\neg p$	*p* is equivalent to the negation of not *p*
Transposition	$(p \to q) \vdash (\neg q \to \neg p)$	If *p* then *q* is equiv. to if not *q* then not *p*
Material Implication	$(p \to q) \vdash (\neg p \vee q)$	If *p* then *q* is equiv. to not *p* or *q*
Material Equivalence (1)	$(p \leftrightarrow q) \vdash ((p \to q) \wedge (q \to p))$	(*p* iff *q*) is equiv. to (if *p* is true then *q* is true) and (if *q* is true then *p* is true)
Material Equivalence (2)	$(p \leftrightarrow q) \vdash ((p \wedge q) \vee (\neg p \wedge \neg q))$	(*p* iff *q*) is equiv. to either (*p* and *q* are true) or (both *p* and *q* are false)
Material Equivalence (3)	$(p \leftrightarrow q) \vdash ((p \vee \neg q) \wedge (\neg p \vee q))$	(*p* iff *q*) is equiv to., both (*p* or not *q* is true)

		and (not p or q is true)
Exportation[9]	$((p \wedge q) \to r) \vdash (p \to (q \to r))$	from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)
Importation	$(p \to (q \to r)) \vdash ((p \wedge q) \to r)$	If p then (if q then r) is equivalent to if p and q then r
Tautology (1)	$p \vdash (p \vee p)$	p is true is equiv. to p is true or p is true
Tautology (2)	$p \vdash (p \wedge p)$	p is true is equiv. to p is true and p is true
Tertium non datur (Law of Excluded Middle)	$\vdash (p \vee \neg p)$	p or not p is true
Law of Non-Contradiction	$\vdash \neg(p \wedge \neg p)$	p and not p is false, is a true statement