MATH 8: HANDOUT 15

EUCLIDEAN GEOMETRY 3: TRIANGLE INEQUALITIES. QUADRILATERALS.

8. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 9). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

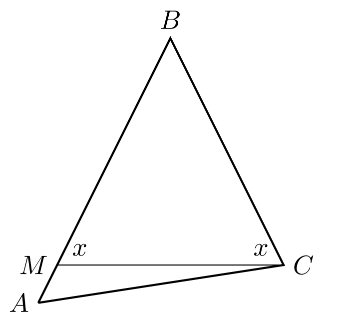
Theorem 11. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$ (these two angles are denoted by x in the figure.) On one hand, $m\angle C > x$ (this easily follows from Axiom 3). On the other hand, since x is an external angle of $\triangle AMC$, by Problem 6 from Handout 14, we have $x > m\angle A$. These two inequalities imply $m\angle C > m\angle A$, which contradicts what we started with.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction.



□

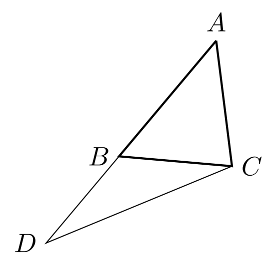
The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 12. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 13 (The triangle inequality). *In $\triangle ABC$, we have $AB + BC > AC$.*

Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 11, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □



9. Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex A is opposite vertex C, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

Definition. A quadrilateral is called

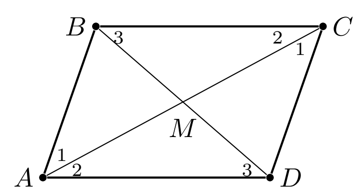
- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length

- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

Theorem 14. *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*



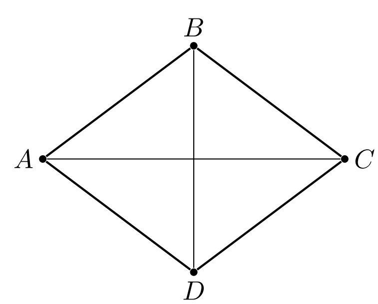
Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. □

Theorem 15. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*

Proof is left to you as a homework exercise.

Theorem 16. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*



Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem 15 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 13 in Assignment Euclidean Geometry 3, it is also the altitude. □

Homework

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

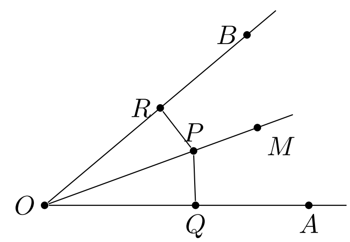
1. (Slant lines and perpendiculars) Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$. Prove that then, for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.

   Note: you can not use the Pythagorean theorem for this, as we haven't yet proved it! Instead, use Theorem 11.
2. (Angle bisector). Define a distance from a point P to line l as the length of the perpendicular from P to l (compare with the previous problem).

Let $\overrightarrow{OM}$ be the angle bisector of $\angle AOB$, i.e. $\angle AOM \cong \angle MOB$.

(a) Let P be any point on $\overrightarrow{OM}$, and PQ, PR – perpendiculars from P to sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Use ASA axiom to prove that triangles $\triangle OPR$, $\triangle OPQ$ are congruent, and deduce from this that distances from P to $\overrightarrow{OA}$, $\overrightarrow{OB}$ are equal.

(b) Prove that conversely, if P is a point inside angle $\angle AOB$, and distances from P to the two sides of the angle are equal, then P must lie on the angle bisector $\overrightarrow{OM}$



These two statements show that the locus of points equidistant from the two sides of an angle is the angle bisector

3. Prove that in any triangle, the three angle bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors – Problem 8 from Handout 14)

4. (Parallelogram) Who doesn't love parallelograms?

(a) Prove Theorem 15.

(b) Prove that if in a quadrilateral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.

5. Prove that in a parallelogram, sum of two adjacent angles is equal to $180°$:

$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180°$$

6. (Rectangle) A quadrilateral is called rectangle if all angles have measure $90°$.

(a) Show that each rectangle is a parallelogram.

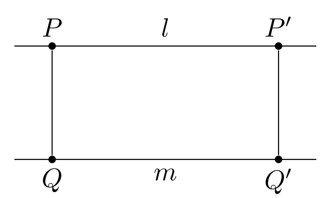
(b) Show that opposite sides of a rectangle are congruent.

(c) Prove that the diagonals of a rectangle are congruent.

(d) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.

7. (Distance between parallel lines)

Let l, m be two parallel lines. Let $P \in l$, $Q \in m$ be two points such that $\overleftrightarrow{PQ} \perp l$ (by Theorem 6, this implies that $\overleftrightarrow{PQ} \perp m$).
Show that then, for any other segment $P'Q'$, with $P' \in l$, $Q' \in m$ and $\overleftrightarrow{P'Q'} \perp l$, we have $PQ = P'Q'$. (This common distance is called the distance between l, m.)



8. The following statements about a parallelogram can be used as its definition, i.e. you can prove any of them from any other. Can you show how?

We have done some of the proofs already. Establish which other statements need to be proven to show the equivalence of all of these statements, and try to prove them. For example, Theorem 15 proves (b) $\Rightarrow$ (a), and Theorem 14 proves (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), and (a) $\Rightarrow$ (d); (e) $\Rightarrow$ (a) is proven in Problem 4b.

(a) Opposite sides are parallel.

(b) Opposite sides are congruent.

(c) Opposite angles are congruent.

(d) Diagonals bisect each other.

(e) One pair of opposite sides is parallel and congruent.