MATH 8: HANDOUT 13
EUCLIDEAN GEOMETRY 1: AXIOMS. PARALLEL LINES

Euclidean geometry tries to describe geometric properties of various figures in the plane. Figures are understood as sets of **points**; we will use capital letters for points and write $P \in m$ for *"point P lies in figure m"*, or *"figure m contains point P"*. The notion of **point** can not be defined: it is so basic that it is impossible to explain it in terms of simpler notions. In addition, there are some other basic notions (**lines**, **distances**, **angle measures**) that can not be defined. Instead, we can state some basic properties of these objects; these basic properties are usually called **postulates** or **axioms of Euclidean geometry**. **All results in Euclidean geometry should be proven by deducing them from the axioms**; justifications *"it is obvious"*, *"it is well-known"*, or *"it is clear from the figure"* are not acceptable!

We allow use of all logical rules. We will also use all the usual properties of real numbers, equations, inequalities, etc.

For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's *Elements*, dated about 300 BC, and used as the standard textbook for the next 2000 years. Nowadays it is available online at `http://math.clarku.edu/~djoyce/java/elements/toc.html`

1. Basic objects

These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points (usually denoted by upper-case letters: A, B, ...)
- Lines (usually denoted by lower-case letters: l, m, ...)
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B. The distance is zero if and only if points coincide.
- Angle measures: for any angle $\angle ABC$, there is a non-negative real number $m\angle ABC$, called the measure of this angle (more on this later).

We will also frequently use words *"between"* when describing relative position of points on a line (as in: A is between B and C) and *"inside"* (as in: point C is inside angle $\angle AOB$). We do not give full list of axioms for these notions; it is possible, but rather boring.

Having these basic notions, we can now define more objects. Namely, we can give definitions of

- interval, or line segment (notation: $\overline{AB}$): set of all points on line $\overleftrightarrow{AB}$ which are between A and B, together with points A and B themselves;
- ray (notation: $\overrightarrow{AB}$): set of all points on the line $\overleftrightarrow{AB}$ which are on the same side of A as B (Note that we have not defined the concept *"on the same side"* but will be using it in the future);
- angle (notation: $\angle AOD$): figure consisting of two rays with a common vertex;
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself (it is a rather convenient convention, which will make our lives easier – the intuition here is that parallel lines have the same "direction").

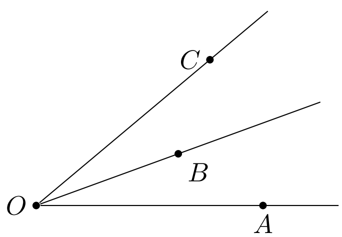
2. First axioms

After we introduced some objects, including undefined ones, we need to have statements (*axioms*) that describe their properties. Of course, the lack of definition for undefined objects makes such properties impossible to prove. The goal here is to state the *minimal number* of such properties that we take for granted, just enough to be able to prove or derive harder and more complicated statements. Here are the first few axioms:

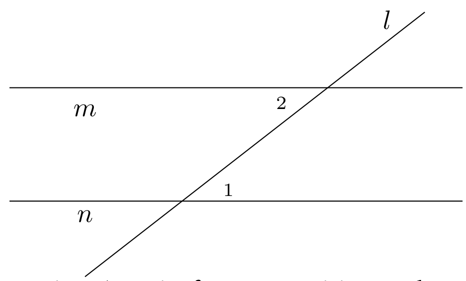
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

3. First theorems

Now we can proceed with proving some results based on the axioms above.

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Proof. Proof by contradiction: Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more than one point) must be false. $\square$

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Proof. Here we have to prove two things: the existence of a parallel line through the given point not on the given line, and its uniqueness. Below we provide a sketch of the proof – please fill in the details and draw a diagram at home!

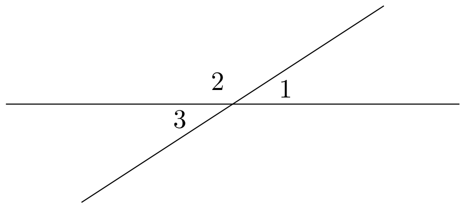
Existence: Let m be any line that goes through P and intersect l at point O. Let A be a point on the line l. Then we can measure the angle $\angle POA$. Now, let PB be such that $m\angle BPO = m\angle POA$ and B is on the other side of m than A. In this case, by Axiom 4, $\overleftrightarrow{AB} \parallel l$.

Uniqueness: Imagine that there are two lines m, n that are parallel to l and go through P. Take a line k that goes through P and intersects l in point O. Let A be a point on line l distinct from O, and B, C — points on lines m and n respectively on the other side of line k than A. Since both m, n are parallel to l, we can see that $m\angle AOP = m\angle BPO = m\angle CPO$ – but that would mean that lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{CP}$ are the same — contradiction to our assumption that there are two such lines. $\square$

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Proof. Assume that l and n are not parallel and intersect at point P. But then it appears that there are two lines that are parallel to m are go through point P — contradiction with Theorem 2. $\square$

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Proof. Left as a homework exercise. □

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Proof. Left as a homework exercise. □

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Proof. Left as a homework exercise. □

HOMEWORK

All problems below are important — please try to finish them all! Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms. Each step of your proof must be based on some previous, already proven, statement or an axiom.

1. It is important that you know some geometry notation.
  (a) What does the symbol $\parallel$ mean? How do you pronounce it? How would you read "$a \parallel b$"?
  (b) What does the symbol $\perp$ mean? How would you say "$a \perp b$"?
  (c) Suppose you have two points X and Y. What is the difference between $\overline{XY}$, $\overleftrightarrow{XY}$, $\overrightarrow{XY}$? What are each of these things called?
  (d) Given three points E, F, G, what does $EF + FG$ mean?
  (e) Given four points A, B, C, D, what does $m\angle ADC + m\angle BDC$ mean? If I tell you $m\angle ADC + m\angle BDC = 180°$, does that tell you any information about $m\angle ADC$ or $m\angle BDC$?
  (f) What does the symbol $\triangle$ mean? For example, if A and B and C are points, what is $\triangle ABC$?

2. (a) What is a proof? Give an example. Can you come up with an example that is not about geometry?
  (b) What is an axiom? Give an example. Can you come up with an example that is not about geometry?

3. In this problem, you will make diagrams. Part of the purpose of this exercise is so that, when you think about geometry, the pictures in your notes or in your mind aren't all just the diagrams I draw out for you in class or on classwork sheets. You have to be able to draw or visualize configurations of lines other than the way they're set up in axiom 4, for example.
  (a) Given lines a, b, c, is it possible that $a \parallel b$ and $\neg(b \parallel c)$ but $a \parallel c$? Draw a diagram and then explain your reasoning on how to answer this question. ("explain" means, of course, in writing.)
  (b) Suppose we have parallel lines l, m. Let A, B, C be points on l, with B between A, C. Let X, Y, Z be points on m, with Y between X, Z. Is it possible for lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, $\overleftrightarrow{CZ}$ to all intersect at one point? Draw a diagram of what this might look like.

(c) Consider the diagram you drew in the previous part, with the lines l, m and the six points, and the three cross-lines that intersect at a point. Now consider the lines $\overleftrightarrow{AZ}$, $\overleftrightarrow{CX}$. Do these two lines intersect at a point on $\overleftrightarrow{BY}$? Draw a diagram where this *is* the case, and then draw a second diagram where this *is not* the case.

(d) Draw a rectangle that's not a square, and draw it so that one of the bases is horizontal. Then draw one of the rectangle's diagonals. Notice that, of the two right angles formed at the rectangle's base, the rectangle's diagonal splits one of those angles into two smaller angles. Which of the two angles is bigger - the one below the diagonal, or the one above the diagonal? Draw a second rectangle where the opposite relation holds true (for example, if the lower angle was bigger in your first rectangle, draw a second rectangle where the lower angle split by the diagonal is smaller).

4. Can you formulate Axiom 4 without referring to the picture (i.e. without using any statement such as "angles $\angle 1$, $\angle 2$ are as shown in the figure below"? You will have to introduce a number of points and have very clear notations.

5. The following logic and geometric statements come in equivalent pairs. Each logic statement has exactly one geometric statement that is equivalent to it. Match these statements into their equivalent pairs, with an explanation of why the pairs you chose are equivalent. [**Note**: the quantifier $\exists!$ stands for "there exists a unique...", and $\emptyset$ is an empty set.]

Geometric statements:

(a) For any two distinct points there is a unique line containing these points.

(b) Given a line and a point not on the line there exists a unique line though the given point that is parallel to the given line.

(c) If two lines are parallel and another line intersects one of them, then it intersects the other one as well.

(d) If two lines are parallel to the same line, then they are parallel to each other

Logic statements:

(a) $\forall l \; \forall m$ such that $l \parallel m \; [\forall n \; (n \cap l \neq \emptyset \rightarrow n \cap m \neq \emptyset)]$

(b) $\forall A \; \forall B$ such that $A \neq B \quad [\exists! \; l \; (A \in l \wedge B \in l)]$

(c) $\forall l \; \forall m \; [(\exists n$ such that $n \parallel l \wedge n \parallel m) \rightarrow (l \parallel m)]$

(d) $\forall l \; \forall A$ such that $A \notin l \quad [\exists! \; m \; (A \in m \wedge m \parallel l)]$

6. (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.

(a) Study the proof of Theorem 2 and draw a diagram that illustrates it.

(b) Study the proof of Theorem 3.

(c) Prove Theorem 5.

(d) Prove Theorem 6.

(e) Prove Theorem 7.