Main formulas of combinatorics

We continued studying the numbers $\binom{n}{k}$ from the Pascal triangle – during the last class we figure out that thee numbers answer many various questions:

$$\binom{n}{k} = \text{The number of paths on the chessboard going } k \text{ steps up and } n - k \text{ to the right}$$
$$= \text{The number of words that can be written using } k \text{ zeros and } n - k \text{ ones}$$
$$= \text{The number of ways to choose } k \text{ items out of } n \text{ if the } \textbf{order does not matter}$$

There exists an explicit formula for them

(1)
$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!}$$

It is possible to understand why this formula works: the numerator $n(n-1)\dots(n-k+1)$ is a number of permutations of k elements out of n — $_nP_k$ (when the order matters). Now, to get the formula for $\binom{n}{k}$, we need to divide the number of permutations by the number of different reorderings of k elements (remember how we divided $n(n-1)$ by 2 when we counted all handshakes in a group of n people?)

Binomial formula

These numbers have one more important application:

(2)
$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n}b^n$$

The general term in this formula looks like $\binom{n}{k} \cdot a^{n-k}b^k$. For example, for $n = 3$ we get

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

(compare with the 3rd row of Pascal's triangle)

This formula is called the **binomial formula**; we discussed its proof today.

Problems

In all the problems, you can write your answer as a combination of factorials, $_nC_k$, and other arithmetic – you do not have to do the computations. As usual, please write your reasoning, not just the answers!

1. Use the binomial formula to expand the following expressions:
    (a) $(x-y)^3$
    (b) $(a+3b)^3$
    (c) $(2x+y)^5$
    (d) $(x+2y)^5$

2. Find the coefficient of x^8 in the expansion of $(2x+3)^{14}$

3. Compute $(1+\sqrt{3})^6 + (1-\sqrt{3})^6$

4. Compute $(x+2y)^6 - (x-2y)^6$

5. Show that $(1+\sqrt{3})^{12} + (1-\sqrt{3})^{12}$ is integer.

6. Deduce that Pascal's triangle is symmetric, i.e. $\binom{n}{k} = \binom{n}{n-k}$ in two ways:
    (a) Using the binomial formula for $(x+y)^n$ and $(y+x)^n$.
    (b) Using formula (1).

7. Use the binomial formula to compute
   (a) Sum of all numbers in the n-th row of Pascal's triangle. [Hint: take $a = b = 1$ in the binomial formula.]
   (b) Alternating sum of all numbers in the n-th row of Pascal's triangle: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots$.
       Can you find a way of answerign thsi question without using the binomial formula?

8. Let p be prime.
   (a) Show that each of the binomial coefficients $\binom{p}{k}$, $1 \le k \le p - 1$, is divisible by p.
   (b) Show that if a, b are integer, then $(a + b)^p - a^p - b^p$ is divisible by p.

*9. Long ago, the four nations decided to hold a relay race competition. Forty-eight people signed up, twelve from each of four element-nations: Water, Earth, Fire, Air; however a relay run consists of four people, so only sixteen of those people can compete.
   (a) Given that each nation must select four people to form a team, how many ways can this be done?
   (b) Now consider they run the competition slightly differently: teams will consist of one person from each nation, and four teams will be chosen. How many ways can this be done?

10. [Some of you may have seen this — but not all of you. You can use a calculator (or Wolfram Alpha) for this problem.]
   (a) Given a group of 25 people, we ask each of them to choose a day of the year (non-leap, so there are 365 possible days). How many possible combinations can we get? [Order matters: it is important who has chosen which date]
   (b) The same question, but now we additionally require that all chosen dates be different.
   (c) In a group of 25 people, what are the chances that no two of them have their birthday on the same day? Conversely, what is the chance that at least two people have the same birthday?