# MATH 8: EUCLIDEAN GEOMETRY 5

FEB 7, 2021

We start by repeating some of the material from last week, which we didn't have time to cover properly in class.

## Circles

**Definition.** A circle with center $O$ and radius $r > 0$ is the set of all points $P$ in the plane such that $OP = r$.

Traditionally, one denotes circles by Greek letters: $\lambda, \omega \dots$.
Given a circle $\lambda$ with center $O$,

- A radius is any line segment from $O$ to a point $A$ on $\lambda$,
- A chord is any line segment between distinct points $A$, $B$ on $\lambda$,
- A diameter is a chord that passes through $O$,

Recall that by Theorem 16, if $O$ is equidistant from points $A, B$, then $O$ must lie on the perpendicualr bisector of $AB$. We can restate this result as follows.

**Theorem 27.** *If $AB$ is a chord of circle $\lambda$, then the center $O$ of this circle lies on the perpendicular bisector of $AB$.*

## Relative positions of lines and circles

**Theorem 28.** *Let $\lambda$ be a circle of radius $r$ with center at $O$ and let $l$ be a line. Let $d$ be the distance from $O$ to $l$, i.e. the length of the perpendicular $OP$ from $O$ to $l$. Then:*

- *If $d > r$, then $\lambda$ and $l$ do not intersect.*
- *If $d = r$, then $\lambda$ intersects $l$ at exactly one point $P$, the base of the perpendicular from $O$ to $l$. In this case, we say that $l$ is tangent to $\lambda$ at $P$.*
- *If $d < r$, then $\lambda$ intersects $l$ at two distinct points.*

*Proof.* First two parts easily follow from Theorem 14: slant line is longer than the perpendicular.

For the last part, it is easy to show that $\lambda$ can not intersect $l$ at more than 2 points (see problem 1 of previous homework). Proving that it does intersect $l$ at two points is very hard and requires deep results about real numbers. This proof will not be given here. □

Note that it follows from the definition that a tangent line is perpendicular to the radius $OP$ at point of tangency. Converse is also true.

**Theorem 29.** *Let $\lambda$ be a circle with center $O$, and let $l$ be a line through a point $A$ on $\lambda$. Then $l$ is tangent to $\lambda$ if and only if $l \perp \overleftrightarrow{OA}$*

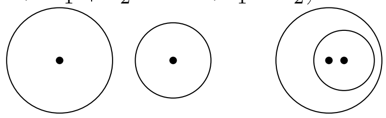
*Proof.* By definition, if $l$ is the tangent line to $\lambda$, then it has only one common point with $\lambda$, and this point is the base of the perpendicular from $O$ to $l$; thus, $OA$ is the perpendicular to $l$.

Conversely, if $OA \perp l$, it means that the distance from $l$ to $O$ is equal to the radius (both are given by $OA$), so $l$ is tangent to $\lambda$. □

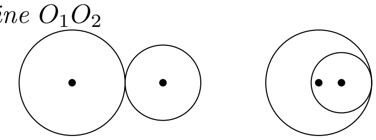
Similar results hold for relative position of a pair of circles. We will only give part of the statement.

**Theorem 30.** *Let $\lambda_1, \lambda_2$ be two circles, with centers $O_1, O_2$ and radiuses $r_1, r_2$ respectively; assume that $r_1 \geq r_2$. Let $d = O_1O_2$ be the distance between the centers of the two circles.*

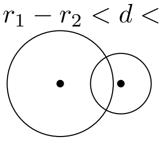
- *If $d > r_1 + r_2$ or $d < r_1 - r_2$, then these two circles do not intersect.*



- *If $d = r_1 + r_2$ or $d = r_1 - r_2$ then these two circles have a unique common point, which lies on the line $O_1O_2$*



- *If $r_1 - r_2 < d < r_1 + r_2$, then the two circles intersect at exactly two points.*



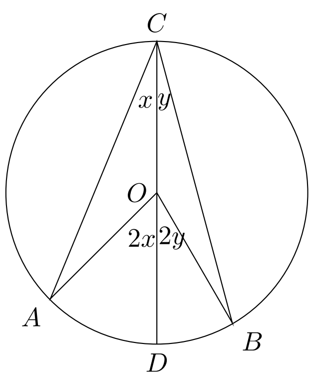
We skip the proof.

**Definition.** Two circles are called tangent if they intersect at exactly one point.

### Arcs and Angles

Consider a circle $\lambda$ with center $O$, and an angle formed by two rays from $O$. Then these two rays intersect the circle at points $A$, $B$, and the portion of the circle contained inside this angle is called the arc subtended by $\angle AOB$. We will sometimes use the notation $\overset{\frown}{AB}$. We define the measure of the arc as the measure of the corresponding central angle: $\overset{\frown}{AB} = m\angle AOB$.

**Theorem 31.** *Let $A$, $B$, $C$ be on circle $\lambda$ with center $O$. Then $m\angle ACB = \frac{1}{2}\overset{\frown}{AB}$. The angle $\angle ACB$ is said to be inscribed in $\lambda$.*




*Proof.* There are actually a few cases to consider here, since $C$ may be positioned such that $O$ is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.
*Case 1.* Draw diameter $\overline{CD}$. Let $x = m\angle ACD$, $y = m\angle BCD$, so that $m\angle ACB = x + y$.
Since $\overline{OC}$ is a radius of $\lambda$, we have that $\triangle AOC$ is isosceles triangle, thus $m\angle A = x$. Therefore, $m\angle AOD = 2x$, as it is the external angle of $\triangle AOC$. Similarly, $m\angle BOD = 2y$. Thus, $\overset{\frown}{AB} = \overset{\frown}{AD} + \overset{\frown}{DB} = 2x + 2y$. $\square$

This theorem has a converse, which essentially says that **all** points $C$ forming a given angle $\angle ACB$ with gven points $A, B$ must lie on a circle containing points $A, B$. Exact statement is given in the homework (see problem 4).

As an immediate corollary, we get the following result:

**Theorem 32.** *Let $\lambda$ be a circle with diameter $AB$. Then for any point $C$ on this circle other than $A, B$, the angle $\angle ACB$ is the right angle. Conversely, if a point $C$ is such that $\angle ACB$ is the right angle, then $C$ must lie on the circle $\lambda$.*

## Homework

**1.** Show that if a circle $\omega$ is tangent to both sides of the angle $\angle ABC$, then the center of that circle must lie on the angle bisector. [Hint: this center is equidistant from the two sides of the circle.] Show that conversely, given a point $O$ on the angle bisector, there exists a circle with center at this point which is tangent to both sides fo the angle.

**2.** Use the previous problem to show that for any triangle, there is a unique circle that is tangent to all three sides (inscribed circle). [Hint: see Theorem 19 in our *Summary of results*.]

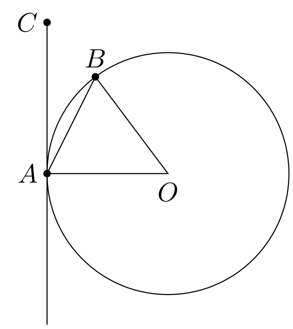
**3.** Given a circle $\lambda$ with center $A$ and a point $B$ outside this circle, construct the tangent line $l$ from $B$ to $\lambda$ using straightedge and compass. How many solutions does this problem have?

[Hint: let $P$ be the tangency point (which we haven't contructed yet). Then by Theorem 29, $\angle APB$ is a right angle. Thus, by Theorem 32, it must lie on a circle with diameter $OP$]

**4.** (Angle Theorems) Let's study Theorem 31 in a bit more detail!

(a) Prove the converse of Theorem 31: namely, if $\lambda$ is a circle centered at $O$ and $A$, $B$, are on $\lambda$, and there is a point $C$ such that $m\angle ACB = \frac{1}{2} m\angle AOB$, then $C$ lies on $\lambda$. [Hint: let $C'$ be the point where line $AC$ intersects $\lambda$. Show that then, $m\angle ACB = m\angle AC'B$, and show that this implies $C = C'$.]

(b) Let $A$, $B$ be on circle $\lambda$ centered at $O$ and $m$ the tangent to $\lambda$ at $A$, as shown on the right. Let $C$ be on $m$ such that $C$ is on the same side of $\overleftrightarrow{OA}$ as $B$. Prove that $m\angle BAC = \frac{1}{2} m\angle BOA$. [Hint: extend $\overline{OA}$ to intersect $\lambda$ at point $D$ so that $\overline{AD}$ is a diameter of $\lambda$. What arc does $\angle DAB$ subtend?]

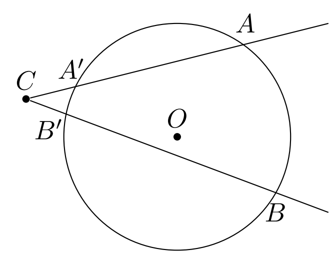



**5.** Here is a modification of Theorem 31.

Consider a circle $\lambda$ and an angle whose vertex $C$ is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\overparen{AB}$ and $\overparen{A'B'}$.

Prove that in this case, $m\angle C = \frac{1}{2}(\overparen{AB} - \overparen{A'B'})$.

[Hint: draw line $AB'$ and find first the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]




**6.** Can you suggest and prove an analog of the previous problem, but when the point $C$ is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vetical angles)?