MATH 8: EUCLIDEAN GEOMETRY 4

JAN 31, 2021

Circles

Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $OP = r$.

Traditionally, one denotes circles by Greek letters: $\lambda, \omega \dots$.
Given a circle λ with center O,

- A radius is any line segment from O to a point A on λ,
- A chord is any line segment between distinct points A, B on λ,
- A diameter is a chord that passes through O,

Recall that by Theorem 16, if O is equidistant from points A, B, then O must lie on the perpendicualr bisector of AB. We can restate this result as follows.

Theorem 27. *If AB is a chord of circle λ, then the center O of this circle lies on the perpendicular bisector of AB.*

Relative positions of lines and circles

Theorem 28. *Let λ be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular OP from O to l. Then:*

- *If $d > r$, then λ and l do not intersect.*
- *If $d = r$, then λ intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is tangent to λ at P.*
- *If $d < r$, then λ intersects l at two distinct points.*

Proof. First two parts easily follow from Theorem 14: slant line is longer than the perpendicular.

For the last part, it is easy to show that λ can not intersect l at more than 2 points (see homework problem 1). Proving that it does intersect l at two points is very hard and requires deep results about real numbers. This proof will not be given here. □

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

Theorem 29. *Let λ be a circle with center O, and let l be a line through a point A on λ. Then l is tangent to λ if and only if $l \perp \overleftrightarrow{OA}$*

Proof. By definition, if l is the tangent line to λ, then it has only one common point with λ, and this point is the base of the perpendicular from O to l; thus, OA is the perpendicular to l.

Conversely, if $OA \perp l$, it means that the distance from l to O is equal to the radius (both are given by OA), so l is tangent to λ. □

Similar results hold for relative position of a pair of circles. We will only give part of the statement.

Theorem 30. *Let λ_1, λ_2 be two circles, with centers O_1, O_2 and radiuses r_1, r_2 respectively; assume that $r_1 \geq r_2$. Let $d = O_1O_2$ be the distance between two circles.*

- *If $d > r_1 + r_2$¡ then these two circles do not intersect.*
- *If $d = r_1 + r_2$, then these two circles have a unique common point.*
- *If $r_1 - r_2 < d < r_1 + r_2$, then the two circles intersect at exactly two points.*

We skip the proof; we also leave it to you to try and complete the theorem, explaining what happens when $d = r_1 - r_2$, or $d < r_1 - r_2$.

Definition. Two circles are called tangent if they intersect at exactly one point.

Constructions with straightedge and compass

Large part of classical geometry are geometric constructions: can we construct a figure with given properties? Traditionally, such constructions are done using straight-edge and compass: the straight-edge tool constructs lines and the compass tool constructs circles. More precisely, it means that we allow the following basic operations:

- Draw (construct) a line through two given or previously constructed distinct points. (Recall that by axiom 1, such a line is unique).
- Draw (construct) a circle with center at previously constructed point O and with radius equal to distance between two previously constructed points B, C
- Construct the intersections point(s) of two previusly constructed lines, circles, or a circle and a line

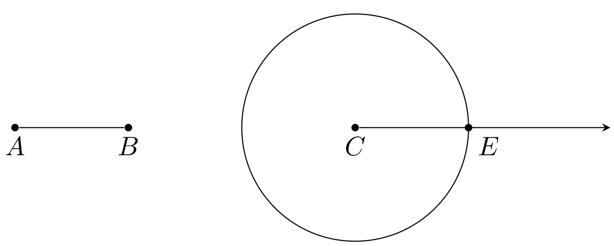
All other constructions (e.g., draw a line parallel to a given one) must be done using these elementary constructions only!!

Constructions of this form have been famous since mathematics in ancient Greece.

Here are some examples of constructions:

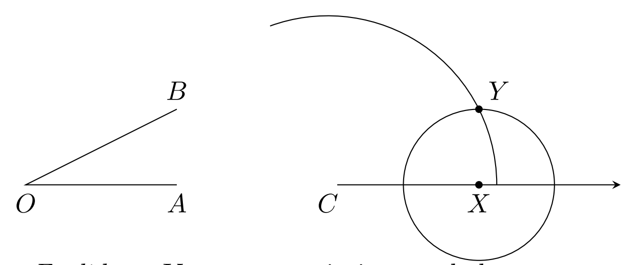
Example 1. Given any line segment $\overline{AB}$ and ray $\overrightarrow{CD}$, one can construct a point E on $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Construction. Construct a circle centered at C with radius AB. Then this circle will intersect $\overrightarrow{CD}$ at the desired point E. □



Example 2. Given angle $\angle AOB$ and ray $\overrightarrow{CD}$, one can construct an angle around $\overrightarrow{CD}$ that is congruent to $\angle AOB$.

Construction. First construct point X on $\overrightarrow{CD}$ such that $CX \cong OA$. Then, construct a circle of radius OB centered at C and a circle of radius AB centered at X. Let Y be the intersection of these circles; then $\triangle XCY \cong \triangle AOB$ by SSS and hence $\angle XCY \cong \angle AOB$. □



A great tool to learn these constructions is an app *Euclidea*. You can use it in a web browser at `http://euclidea.xyz`, or install it on your phone or tablet (it is available both for iOS and Android).

Note: Euclidea starts with a slightly more restrictive set of tools. Namely, it only allows drawing circles with a given center and passing through a given point; thus, you can not use another segment as radius.

Homework

1. Without using Theorem 28, prove that a circle can not have more than two intersections with a line. [Hint: assume it has three intersection points, and use Theorem 27 to get a contradiction.]

2. Prove that given three points A, B, C not on the same line, there is a unique circle passing through these points. This circle is called the circumscribed circle of $\triangle ABC$. Explain how to construct this circle using ruler and compass.

3. Complete levels α, β in Euclidea.