EUCLIDEAN GEOMETRY: SUMMARY OF RESULTS

UPDATED JAN 24, 2021

Definitions

Undefined objects:

- Points (usually denoted by upper-case letters: A, B,..)
- Lines (usually denoted by lower-case letters: l, m,...)
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B. The distance is zero if and only if points coincide.
- Angle measures: for any angle $\angle ABC$, there is a non-negative real number $m\angle ABC$, called the measure of this angle (more on this later).

Defined objects:

- interval, or line segment (notation: $\overline{AB}$): set of all points on line $\overleftrightarrow{AB}$ which are between A and B, together with points A and B themselves
- ray (notation: $\overrightarrow{AB}$): set of all points on the line $\overleftrightarrow{AB}$ which are on the same side of A as B
- angle (notation: $\angle AOD$): figure consisting of two rays with a common vertex
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself.
- triangle (notation: $\triangle ABC$): a figure consisting of 3 distinct points A, B, C, not one the same line, and three segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ (sides of the triangle).
- isosceles triangle: A triangle $\triangle ABC$ is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

Definitions related to congruence:

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Midpoints, bisectors, medians, altitudes

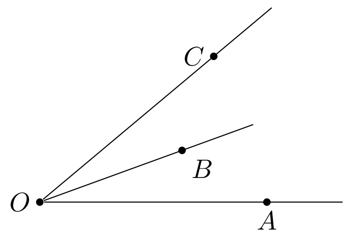
- A midpoint M of a segment $\overline{AB}$ is a point on $\overline{AB}$ such that $AM = MB$.
- A perpendicualr bisector of a segment $\overline{AB}$ is the line l which goes through midpoint of $\overline{AB}$ and is perpendicular to $\overleftrightarrow{AB}$.
- A angle bisector of angle $\angle AOB$ is the ray $\overrightarrow{OM}$ inside this angle such that $\angle AOM \cong \angle MOB$
- A median of a triangle is the line segment connecting a vertex with the midpoint of the opposite side.
- An altitude of the triangle is the line through one of the vertices perpendicular to the opposite side.

Axioms

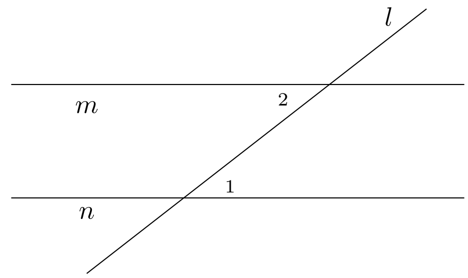
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called* alternate interior angles*). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

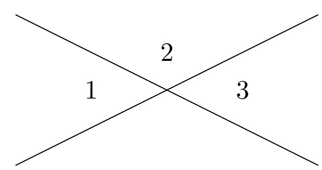
Theorems: Lines and points

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure (such a pair of angles are called* vertical*). Then $m\angle 1 = m\angle 3$.*



Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

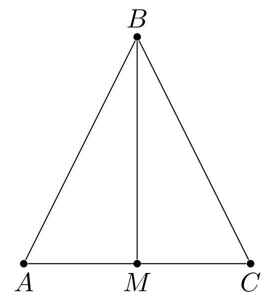
Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Theorems: Triangles

Theorem 8. *In any triangle $\triangle ABC$, the sum of interior angles is equal to $180°$: $m\angle ABC + m\angle BCA + m\angle CAB = 180°$.*

Theorem 9 (Base angles of isosceles triangle)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

Theorem 10. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*



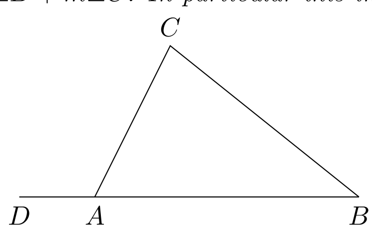
Theorem 11 (Opposite larger angle lies larger side)**.** *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Theorem 12 (Opposite larger side is the larger angle)**.** *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

Theorem 13 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

Theorem 14 (Slant lines and perpendiculars)**.** *Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$. Then for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.*

Theorem 15 (Exterior angle)**.** *Given a triangle $\triangle ABC$, let D be a point on the line AB, so that A is between D and B. (In this situation, angle $\angle DAC$ is called an* exterior angle *of $\triangle ABC$). Then $m\angle DAC = m\angle B + m\angle C$. In particular this implies that $m\angle DAC > m\angle B$, and similarly for $\angle C$.*

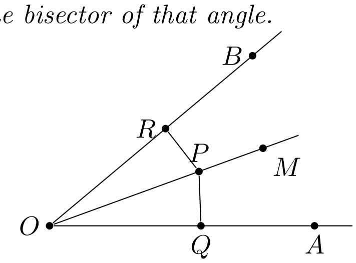


Theorem 16 (Perpendicular bisector as locus of equidistant points)**.** *Given a line segment $\overline{AB}$, a point P is equidistnat from A, B (i.e. $PA = PB$) if and only if P lies on the perpendicualr bisector to AB.*

Theorem 17. *In any triangle $\triangle ABC$, the perpendicular bisectors of the three sides intersect at a single point, and this point is equidistant from all three vertices of the triangle.*

Theorem 18 (Angle bisector as locus of equidistant points)**.** *Define a distance from a point P to line l as the length of the perpendicular from P to l.*

Then a point P inside angle $\angle AOB$ is equidistant from the two sides of the angle if and only if it lies on the bisector of that angle.



Theorem 19. *In any triangle, the three angle bisectors intersect at a single point, and this point is equidistant from the three sides of the triangle.*