MATH 8: EUCLIDEAN GEOMETRY 3

JANUARY 24, 2021

Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex A is opposite vertex C, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

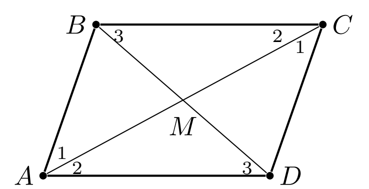
Definition. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

Theorem 20. *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

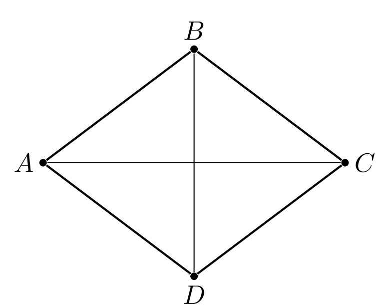


Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. □

Theorem 21. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*

Proof is left to you as a homework exercise.

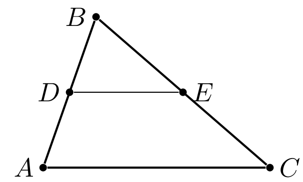


Theorem 22. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem **??** that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 13 in Assignment Euclidean Geometry 3, it is also the altitude. □

Midline of a triangle and trapezoid

Definition. A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two side.



Theorem 23. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

The proof of this theorem is also given as a homework; it is not very easy.

Homework

1. (Parallelogram) Who doesn't love parallelograms?
  (a) Prove Theorem **??**
  (b) Prove that if in a quadrilateral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.

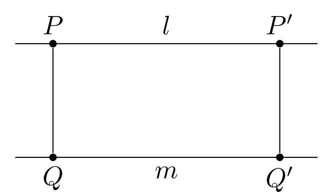
2. Prove that in a parallelogram, sum of two adjacent angles is equal to 180°:

$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180^\circ$$

3. (Rectangle) A quadrilateral is called rectangle if all angles have measure 90°.
  (a) Show that each rectangle is a parallelogram.
  (b) Show that opposite sides of a rectangle are congruent.
  (c) Prove that the diagonals of a rectangle are congruent.
  (d) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.

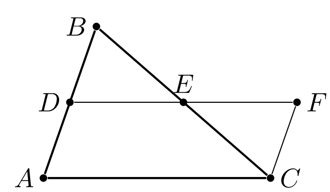
4. (Distance between parallel lines)

Let l, m be two parallel lines. Let $P \in l$, $Q \in m$ be two points such that $\overleftrightarrow{PQ} \perp l$ (by Theorem 6, this implies that $\overleftrightarrow{PQ} \perp m$).
Show that then, for any other segment $P'Q'$, with $P' \in l$, $Q' \in m$ and $\overleftrightarrow{P'Q'} \perp l$, we have $PQ = P'Q'$. (This common distance is called the distance between l, m.)



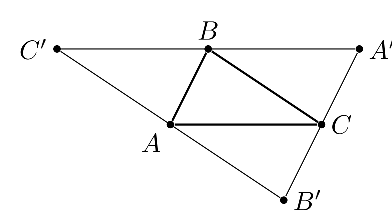
5. (Triangle Midline) Prove Theorem **??** by completing the steps below.

Continue line DE and mark on it point F such that $DE = EF$.
  (a) Prove that $\triangle DEB \cong \triangle FEC$
  (b) Prove that $ADFC$ is a parallelogram (hint: use alternate interior angles!)
  (c) Prove that $DE = \frac{1}{2}AC$



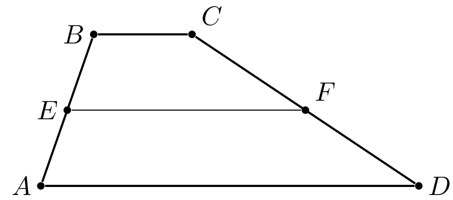
6. Show that if we mark midpoints of each of the three sides of a triangle, and connect these points, the resulting segments will divide the original triangle into four triangles, all congruent to each other.

7. (Altitudes intersect at single point)
The goal of this problem is to prove that three altitudes of a triangle intersect at a single point. Given a triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the intersection points of these lines by A', B', C' as shown in the figure.
  (a) Prove that $A'B = AC$ (hint: use parallelograms!)
  (b) Show that B is the midpoint of $A'C'$, and similarly for other two vertices.
  (c) Show that altitudes of $\triangle ABC$ are exactly the perpendicular bisectors of sides of $\triangle A'B'c'$.
  (d) Prove that the three altitudes of $\triangle ABC$ intersect at a single point.



8. (Trapezoid Midline)

Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively. Prove that then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.



[Hint: draw through point F a line parallel to AB, as shown in the figure. Prove that this gives a parallelogram, in which points E, F are midpoints of opposite sides.]