MATH 8: ASSIGNMENT 13: EUCLIDEAN GEOMETRY 1

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Euclidean geometry tries to describe geometric properties of various figures in the plane. Figures are understood as sets of points; we will use capital letters for points and write $P \in m$ for "point P lies in figure m", or "figure m contains point P". The notion of "point" can not be defined: it is so basic that it is impossible to explain it in terms of simpler notions. In addition, there are some other basic notions (lines, distances, angles) that can not be defined. Instead, we can state some basic properties of these objects; these basic properties are usually called "postulates" or "axioms of Euclidean geometry". **All results in Euclidean geometry should be proved by deducing them from the axioms**; justifications "it is obvious", "it is well-known", or "it is clear from the figure" are not acceptable.

We allow use of all logical rules. We will also use all the usual properties of real numbers, equations, inequalities, etc.

For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's *Elements*, dated about 300 BC, and used as the standard textbook for the next 2000 years. Nowadays it is avaialable online at `http://math.clarku.edu/~djoyce/java/elements/toc.html`

1. Basic objects

These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points (usually denoted by upper-case letters: A, B,..)
- Lines (usually denoted by lower-case letters: l, m,...)
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B. The distance is zero if and only if points coincide.
- Angle measures: for any angle $\angle ABC$, there is a non-negative real number $m\angle ABC$, called the measure of this angle (more on this later).

We will also frequently use words "between" when describing relative position of points on a line (as in: A is between B and C) and "inside" (as in: point C is inside angle $\angle AOB$). We do not give full list of axioms for these notions; it is possible, but rather boring.

Having these basic notions, we can now define more objects. Namely, we can give definitions of

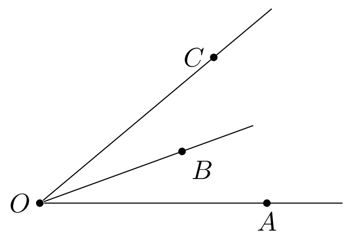
- interval, or line segment (notation: $\overline{AB}$): set of all points on line $\overleftrightarrow{AB}$ which are between A and B, together with points A and B themselves
- ray (notation: $\overrightarrow{AB}$): set of all points on the line $\overleftrightarrow{AB}$ which are on the same side of A as B
- angle (notation: $\angle AOD$): figure consisting of two rays with a common vertex
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself.

2. First postulates

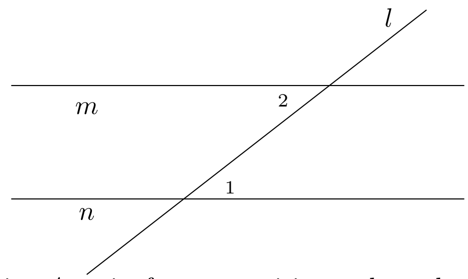
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called* alternate interior angles*). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angels: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

3. First theorems

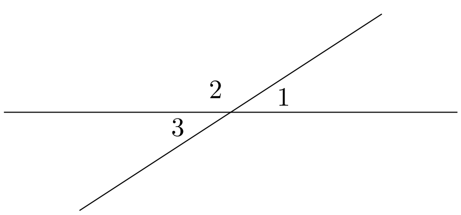
Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Proof. Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom **??**. Thus, our assumption (that l, m intersect at more then one point) must be false. □

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called* vertical*). Then $m\angle 1 = m\angle 3$.*



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom **??**, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*

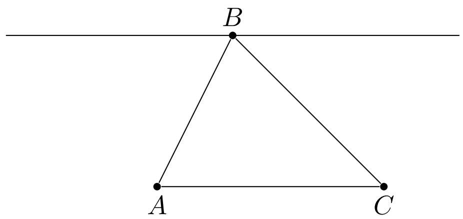
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

4. Triangles

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a* triangle*, written $\triangle ABC$. The three respective angles are called the triangle's* interior angles.*)*

Proof. The proof is based on the figure below and use of Alternate Interior Angles axiom. Details are left to you as a homework.



□

5. Congruence

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

6. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

7. Homework

Note that you may use all results that are presented in the previous sections. This means that you may use Theorem 3, for example, if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

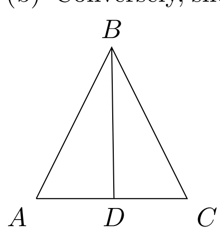
1. (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.
  (a) Prove Theorem 2.
  (b) Prove Theorem 3. [Hint: assume that l and n are not parallel; then they must intersect at some point P.]
  (c) Prove Theorem 5.
  (d) Prove Theorem 6.
  (e) Prove Theorem 7.

2. Complete the proof of Theorem 8, about sum of angles of a triangle.

3. What is the sum of angles of a quadrilateral? of a pentagon?

4. Notice that SSA and AAA are not listed as congruence rules.
  (a) Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.
  (b) Describe a pair of triangles that have three congruent angles but are not congruent triangles.

5. (Isosceles Triangles) A triangle in which two sides are congruent is called *isosceles.* Such triangles have many special properties.
  (a) Let $\triangle ABC$ be an isosceles triangle, with $\overline{AB} \cong \overline{BC}$. Suppose D is a point on $\overline{AC}$ such that $\overline{AD} \cong \overline{DC}$ (such point is called *midpoint* of the segment). Prove that then, $\triangle BD \cong \triangle CBD$ and deduce from this that $\angle DBA \cong \angle DBC$, and $\angle A \cong \angle C$. What can we say about $\angle ADB$?
  (b) Conversely, show that if $\triangle ABC$ is such that $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{BC}$.



6. Prove that the following two properties of a triangle are equivalent:
  (a) All sides have the same length
  (b) All angles are 60°.

  A triangle satisfying these properties is called *equilateral.*