

MATH 8
ASSIGNMENT 10: PROOFS CONTINUED
DEC 6, 2020

COMMONLY USED LAWS OF LOGIC

- Given $A \implies B$ and A , we can conclude B (Modus Ponens)
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C .]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ (Modus Tollens)
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ (De Morgan Law)
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ (law of contrapositive)

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

COMMON METHODS OF PROOF

Proof by cases.

Example: Prove that for any integer n , the number $n(n+1)$ is even.

Proof. If n is integer, it is even or odd. If n is even, then $n(n+1)$ is even (a multiple of even is always even). If n is odd, then $n+1$ is even and thus $n(n+1)$ is even by same reasoning.

Thus, in all cases $n(n+1)$ is even. □

General scheme

Given

$$\begin{aligned} &A_1 \vee A_2 \\ &A_1 \implies B \\ &A_2 \implies B \end{aligned}$$

we can conclude that B is true.

You can have more than two cases.

Note: it is important to verify that the cases you consider cover all possibilities (i.e. tht at least one of the statements A_1, A_2 is always true).

Conditional proof.

Example: Prove that if n is even, then n^2 is even.

Proof. Assume that n is even. Then $n^2 = n * n$ is also even, since a multiple of even is even. □

General scheme

To prove $A \implies B$, we can

- Assume A
- Give a proof of B (in the proof, we can use that A is true).

This proves $A \implies B$ (without any assumptions).

Proof by contradiction.

Example: Prove that if x is a real root of polynomial $p(x) = 10x^3 + 2x + 15$, then x must be negative.

Proof. Assume that x is not negative, i.e. $x \geq 0$. Then $p(x) = 10x^3 + 2x + 15 \geq 15$, which contradicts the fact that x is a root of $p(x)$. Thus, our assumption can not be true, so x must be negative. □

General scheme

To prove that A is true, assume A is false, and derive a contradiction. This proves that A must be true.

PROBLEMS

1. The following statement is sometimes written on highway trucks:

If you can't see my windows, I can't see you.

Can you write an equivalent statement without using word "not" (or its variations such as "can't").

2. Consider the following statement:

You can't be happy unless you have a clear conscience.

Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter H for "you are happy" and C for "you have a clear conscience".

Note: proving this statement is not part of the assignment :).

3. Here is another one of Lewis Carroll's puzzles. As before, a)write the obvious conclusion from given statements; and b)justify the conclusion, by writing a chain of arguments which leads to it.

- No one subscribes to the *Times*, unless he is well educated.
- No hedgehogs can read.
- Those who cannot read are not well educated.

It may be helpful to write each of these as a statement about some particular being X , e.g. "If X is a hedgehog, then X can't read."

4. Prove that for any integer number n , the number $n(n+1)(2n+1)$ is divisible by 3. Is it true that such a number must also be divisible by 6?

You can use without proof the fact that any integer can be written in one of the forms $n = 3k$ or $n = 3k + 1$ or $n = 3k + 2$, for some integer k .

5. You are given the following statements:

$$A \wedge B \implies C$$

$$B \vee D$$

$$C \vee \neg D$$

Using this, prove $A \implies C$.

6. A function $f(x)$ is called *monotonic* if $(x_1 < x_2) \implies (f(x_1) < f(x_2))$. Prove that a monotonic function can't have more than one root. [Hint: use assume that it has at least two distinct roots and derive a contradiction.]
7. Prove by contradiction that there does not exist a smallest positive real number.