MATH 8: EUCLIDEAN GEOMETRY 5

FEB 28, 2021

Circles

Given a circle λ with center O,

- A radius is any line segment from O to a point A on λ,
- A chord is any line segment between distinct points A, B on λ,
- A diameter is a chord that passes through O,
- A tangent line is a line that intersects the circle exactly once; if the intersection point is A, the tangent is said to be the tangent through A.

Moreover, we say that two circles are tangent if they intersect at exactly one point.

Theorem 20. *Let A be a point on circle λ centered at O, and m a line through A. Then m is tangent to λ if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to λ at A.*

Proof. First we prove (m is tangent to λ) $\implies$ ($m \perp \overline{OA}$). Suppose m is tangent to λ at A but not perpendicular to $\overline{OA}$. Let $\overline{OB}$ be the perpendicular to m through O, with B on m. Construct point C on m such that $BA = BC$; then we have that $\triangle OBA \cong \triangle OBC$ by SAS, using $OB = OB$, $\angle OBA = \angle OBC = 90°$, and $BA = BC$. Therefore $OC = OA$ and hence C is on λ. But this means that m intersects λ at two points, which is a contradiction.
Now we prove ($m \perp \overline{OA}$) $\implies$ (m is tangent to λ). Suppose m passes through A on λ such that $m \perp \overline{OA}$. If m also passed through B on λ, then $\triangle AOB$ would be an isosceles triangle since $\overline{AO}$, $\overline{BO}$ are radii of λ. Therefore $\angle ABO = \angle BAO = 90°$, i.e. $\triangle AOB$ is a triangle with two right angles, which is a contradiction. □

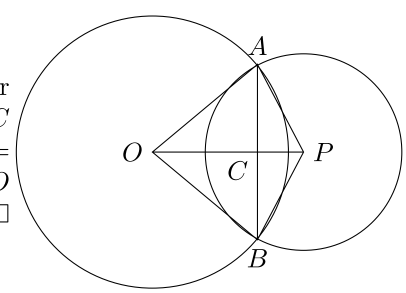
Notice that, given point O and line m, the perpendicular $\overline{OA}$ from O to m (with A on m) is the shortest distance from O to m, therefore the locus of points of distance exactly OA from O should line entirely on one side of m. This is essentially the idea of the above proof.

Theorem 21. *Let $\overline{AB}$ be a chord of circle λ with center O. Then O lies on the perpendicular bisector of $\overline{AB}$. Moreover, if C is on $\overline{AB}$, then C bisects $\overline{AB}$ if and only if $\overline{OC} \perp \overline{AB}$.*

Proof. Let m be the perpendicular bisector of $\overline{AB}$. The center O of λ is equidistant from A, B by the definition of a circle, therefore by Theorem 14, O must be on m. Let m intersect $\overline{AB}$ at D. We then have that D is the midpoint of $\overline{AB}$ and also the foot of the perpendicular from O to $\overline{AB}$.
Then if C bisects $\overline{AB}$, C lies on the perpendicular bisector m of $\overline{AB}$, which passes through O, thus $\overline{OC} \perp \overline{AB}$.
Lastly if $\overline{OC} \perp \overline{AB}$, then because there is only one perpendicular to $\overline{AB}$ through O, we must have $C = D$ and hence C is the midpoint of $\overline{AB}$. □

Theorem 22. *Let λ, ω be circles that intersect at points A, B. Then $\overline{AB} \perp \overline{OP}$.*

Proof. We have that $\triangle AOB$ and $\triangle APB$ are both isosceles, thus their altitudes from O and P respectively both intersect $\overline{AB}$ at the midpoint C of $\overline{AB}$. Then, since $m\angle OCA = m\angle ACP = 90°$, we have that $m\angle OCP = 180°$, i.e. C lies on the line $\overline{OP}$. Since C is the foot of altitudes from O and P, this completes the proof. □



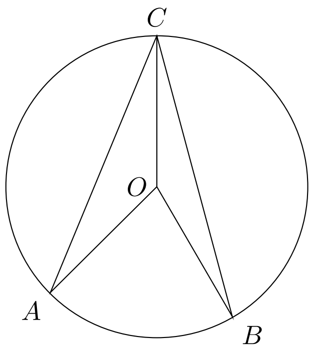
Theorem 23. *Let λ, ω be circles that are both tangent to line m at point A. Then λ, ω are tangent circles.*

Proof. Suppose, by contradiction, that λ, ω intersect at point $B \neq A$. Then $\overline{AB} \perp \overline{OP}$, therefore both $\overline{AB}$ and m are perpendicular to $\overline{OA}$ through A. We must therefore have that B is on m, but m is tangent to λ through A, thus has only one intersection with λ, which is a contradiction. □

Arcs and Angles

Consider a circle λ with center O, and an angle formed by two rays from O. Then these two rays intersect the circle at points A, B, and the portion of the circle contained inside this angle is called the arc subtended by $\angle AOB$.

Theorem 24. *Let A, B, C be on circle λ with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ.*



Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.

Case 1. Draw in segment $\overline{OC}$ and notice that $m\angle AOC + m\angle BOC + m\angle AOB = 360°$. Since $\overline{OC}$ is a radius of λ, we have that $\triangle AOC$ and $\triangle BOC$ are isosceles triangles, thus $m\angle AOC = 180° - 2m\angle OCA$ and $m\angle BOC = 180° - 2m\angle OCB$. Therefore we get $180° - 2m\angle ACO + 180° - 2m\angle BCO + m\angle AOB = 360° \implies m\angle AOB = 2(m\angle ACO + m\angle BCO) \implies m\angle AOB = 2m\angle ACB$. □

As a result of Theorem 24, we get that any triangle $\triangle ABC$ on λ where $\overline{AB}$ is a diameter must be a right triangle, since the angle $\angle ACB$ has half the measure of angle $\angle AOB$, which is 180°.

The idea captured by the concept of an arc and Theorem 24 is that there is a fundamental relationship between angles and arcs of circles, and that the angle 360° can be thought of as a full circle around a point.

Homework

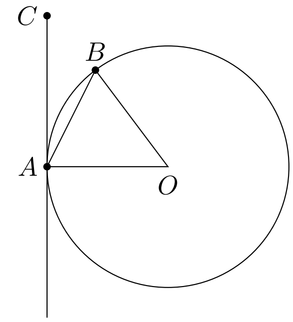
1. Complete the proof of Theorem 24 by proving the cases where O is not inside the angle $\angle ACB$. [Hint: for one of the cases, you may need to write $\angle ACB$ as the difference of two angles.]

2. Given a circle λ with center A, and given a point B outside this circle, construct the tangent line through B tangent to λ.

3. (Angle Theorems) Let's study Theorem 24 in a bit more detail!

(a) Prove the converse of Theorem 24: namely, if λ is a circle centered at O and A, B, are on λ, and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on λ. [Hint: we need to prove that $OC = OA$; consider using a proof by contradiction, using Theorem 11.]

(b) Let A, B be on circle λ centered at O and m the tangent to λ at A, as shown on the right. Let C be on m such that C is on the same side of $\overleftrightarrow{OA}$ as B. Prove that $m\angle BAC = \frac{1}{2}m\angle BOA$. [Hint: extend $\overline{OA}$ to intersect λ at point D so that $\overline{AD}$ is a diameter of λ. What arc does $\angle DAB$ subtend?]

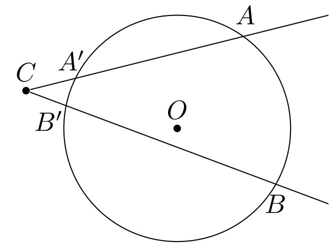


4. Here is a modification of Theorem 24.

Consider a circle λ and an angle whose vertex C is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$.

Prove that in this case, $m\angle C = \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{A'B'})$.

[Hint: draw line AB' and find first the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]



5. Can you suggest and prove an analog of the previous problem, but when the point C is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vetical angles)?

6. Given triangle $\triangle ABC$, complete a straightedge-compass construction of a circle that passes through A, B, C. Deduce that given any three points A, B, C that form a triangle (i.e. are not on the same line), there exists a unique circle through these points.

7. Given lines $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ such that $\overline{AD}$, $\overline{BC}$ intersect at E and $AE = ED$, prove that $BE = EC$.