MATH 8: EUCLIDEAN GEOMETRY 4

FEB 21, 2021

Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 9). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

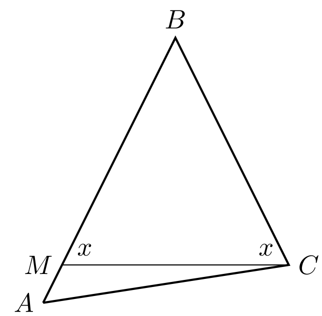
Theorem 17. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$ (these two angles are denoted by x in the figure.) On one hand, $m\angle C > x$ (this easily follows from Axiom 3). On the other hand, since x is an external angle of $\triangle AMC$, by Problem 1 we have $x > m\angle A$. These two inequalities imply $m\angle C > m\angle A$, which contradicts what we started with.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction.



□

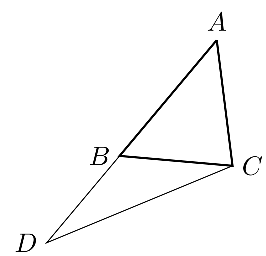
The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 18. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 19 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 17, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □



Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex A is opposite vertex C, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

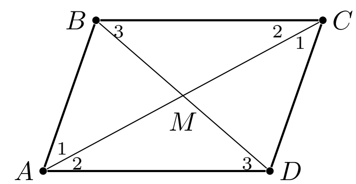
Definition 1. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length

- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

Theorem 20. *Let $ABCD$ be a parallelogram. Then*
- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*



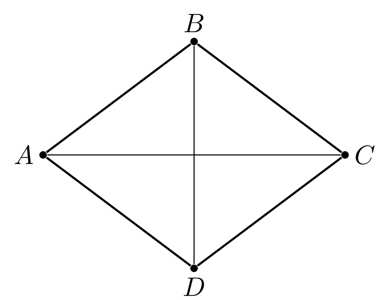
Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. ☐

Theorem 21. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*

Proof is left to you as a homework exercise.

Theorem 22. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*



Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem 21 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 13 in Assignment Euclidean Geometry 3, it is also the altitude. ☐

Homework

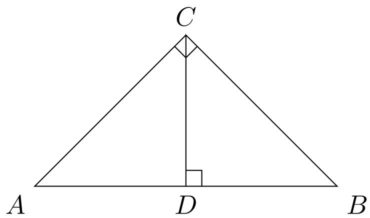
1. (Rectangle) Let $ABCD$ be a rectangle (i.e., all angles have measure 90°).
   - (a) Show that opposite sides of the rectangle are congruent.
   - (b) Prove that the diagonals are congruent.
   - (c) Prove that conversely, if $ABCD$ is instead a parallelogram such that $AC = BD$, then it is a rectangle.

2. (Parallelogram) Who doesn't love parallelograms?
   - (a) Prove Theorem 21
   - (b) Prove that if in a quadrialteral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.

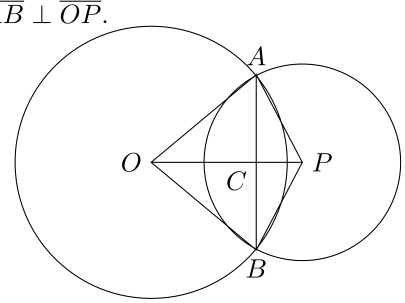
3. (Constructions) Given a triangle $\triangle ABC$, complete (with proof) the following straightedge-compass constructions:
   - (a) Construct the median from A to $\overline{BC}$
   - (b) Construct the altitude from A to $\overline{BC}$
   - (c) Construct the angle bisector from A to $\overline{BC}$
   - (d) Now, for a slightly different exercise, given a circle, complete (with proof) a straightedge-compass construction of the center point of the circle.

4. (AAA) Notice that SSA and AAA are not listed as congruence rules.

(a) Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.
(b) Describe a pair of triangles that have three congruent angles but are not congruent triangles.
(c) In the diagram below, let $m\angle ABC = 45°$. Prove that, in $\triangle ABC$ and $\triangle BCD$, we have $\angle ABC \cong \angle BCD$, $\angle BCA \cong \angle CDB$, and $\angle CAB \cong \angle DBC$. Then notice that $\overline{BC}$ in $\triangle ABC$ is congruent to $\overline{BC}$ in $\triangle BCD$. Can we use the ASA congruence rule to deduce that $\triangle ABC \cong \triangle BCD$?



5. (Circle Cross) Suppose two circles intersect at points A, B, as shown in the diagram. Prove that $\overline{AB} \perp \overline{OP}$.



6. (Quadrilateral Cross) Suppose two congruent line segments have the same midpoint.
(a) Draw a diagram where this is possible, in other words draw a diagram with two congruent but distinct line segments with the same midpoint.
(b) Must the endpoints of these segments form a parallelogram, or is there another possibility? Provide a proof that they must form a parallelogram, or provide a diagram of a counterexample.
(c) Is it possible that the four vertices form a quadrilateral that *is* a parallelogram but *is not* a rhombus? Prove your answer, by proving the statement or providing a counterexample.
(d) Is it possible that the four vertices form a quadrilateral that *is* a parallelogram but *is not* a rectangle? Prove your answer, by proving the statement or providing a counterexample.