MATH 8: EUCLIDEAN GEOMETRY

JANUARY 31, 2021

1. Locus & Circle

Consider some logical property of points - i.e., some statement that can be either true or false for any given point on the plane. Such examples include: the point's distance from some given point X is 1; the point is inside some given triangle $\triangle ABC$; the point is between two given parallel lines l, m. The set of all points for which a property holds true is called the locus of points satisfying the property.

As your first example, the locus of points that are a fixed distance from a given point O is called a circle. The point O is called the center of the circle, and the fixed distance is called the radius distance. A line segment from O to some point on the circle is called a radius of the circle. Note that all radii of a circle are congruent.

Two different circles are congruent if they have the same radius distance; equivalently, if their respective radii are congruent.

Circles are vastly useful objects, and it will be enormously helpful to be comfortable with the concept of what a circle is. Fortunately, circles are familiar to most of us. They do have less obvious properties, but we will prove those later. For now, the basic definition will suffice, as well as the following useful definition related to circles: given a circle with center O, a line segment from O to a point on the circle is called a radius of the circle. Note that all radii of a circle are congruent. Isn't that exciting!

2. Right Angle & Right Triangle

An angle whose measure is $90°$ is called a right angle; a triangle where one of the angles is a right angle is called a right triangle. The side opposite the right angle is called the hypotenuse of the triangle; the other two sides are called the legs.

3. Homework

Note that you may use all results that are presented in previous sections, assignments, and homework problems. This means that you may use Theorem 3, for example, if you find it a useful logical step in your proof.

1. Given a rectangle $ABCD$, let M be the intersection of its diagonals. Prove that M is equidistant from all four vertices (in other words, prove that the distance from M to each of the four vertices is the same).

2. Let A, B be two points on a circle. Prove that the circle's center is on the perpendicular bisector of $\overline{AB}$.

3. Suppose $\triangle ABC$ is a triangle.
   (a) Prove that all three sides of $\triangle ABC$ are congruent if and only if all three angles measure $60°$. Such a triangle is called equilateral.
   (b) Let X be on $\overline{AB}$, Y on $\overline{BC}$, Z on $\overline{AC}$ such that $\overline{AX} \cong \overline{BY} \cong \overline{CZ}$. Prove that if $\triangle ABC$ is equilateral, then $\triangle XYZ$ is equilateral. Make a guess as to whether the converse is true.

4. Review the list of theorems discussed so far in the homework/classwork sheets. Pick one of the theorems (for example, one that you find interesting), and write out a proof of that theorem that uses only previously mentioned theorems and axioms.

5. (a) Suppose we have parallel lines l, m. Let A, B, C be points on l, with B between A, C. Let X, Y, Z be points on m, with Y between X, Z. Is it possible for lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, $\overleftrightarrow{CZ}$ to all intersect at one point? Draw a diagram of what this might look like.
   (b) Consider the diagram you drew in the previous part, with the lines l, m and the six points, and the three cross-lines that intersect at a point. Now consider the lines $\overleftrightarrow{AZ}$, $\overleftrightarrow{CX}$. Do these two lines intersect at a point on $\overleftrightarrow{BY}$? Draw a diagram where this *is* the case, and then draw a second diagram where this *is not* the case.

6. The following logic statements come in equivalent pairs. For each statement on the list, there is exactly one other statement that is equivalent to it. Sort these statements into their equivalent pairs, with an explanation of why the pairs you chose are equivalent.

(a) $\forall x \exists y (x > y)$
(b) $\forall a \forall b (\exists n (a < n < b))$
(c) $\exists x (x^2 > 0 \implies x > 0)$
(d) $\forall e \exists x (e > x)$
(e) $\forall u \forall s (\exists p (u < p < s))$
(f) $\exists x (\exists y (x \implies y))$
(g) $\forall a (\forall b (a \leq b \vee b \leq a))$
(h) $\forall b (\forall a (a \leq b \vee b \leq a))$
(i) $\exists b (b^2 > 0 \implies b > 0)$
(j) $\exists f (\exists g (\neg g \implies \neg f))$

7. A *pentagon* is a figure consisting of five line segments and five angles; a *regular pentagon* is one where all the sides are congruent, and all the angles are congruent (see below).

(a) Find, with proof, the sum of the angle measures of a regular pentagon.
(b) Deduce the angle measure of a single angle of a regular pentagon.
(c) Prove that $\overline{AC} \parallel \overline{DE}$.