MATH 8
LOGIC REVIEW
DEC 13, 2020

Statements and Quantifiers

1. Given statements a and b, if I know that $\neg(a \wedge b)$ is true and I know that a is true, what can I conclude about b?

2. Recall that $\mathbb{Z}$ denotes the set of integers. Prove or disprove the following statement: $\forall p \in \mathbb{Z}$ (p is prime $\implies$ $p+1$ is not a power of 2)

3. Given some object x, let $w(x)$ indicate the statement "x has wheels", and $d(x)$ indicate "x can drive". Let L indicate the universe of objects that exist solely on land. So, for example, let my car be called c_9. Then $w(c_9)$ is true and $d(c_9)$ is true.

(a) Consider the following claim: "any object that exists on land must have wheels in order to drive". How can we represent this as a logical statement?

(b) Suppose I find a wheel factory, which I will call f. This factory has wheels but cannot drive. Is $w(f)$ true? What about $d(f)$?

(c) Given the existence of the factory f, prove or disprove the following statement: $\forall x \in L(w(x) \implies d(x))$.

Logical Equivalence

1. Prove that a positive integer is odd if and only if it can be expressed as a difference of consecutive squares. (Here, *consecutive squares* means the squares of two positive integers y, $y+1$).

2. (a) Given that the product of zero with any integer is zero, and the product of any two nonzero integers is nonzero, prove the following: $\forall x \in \mathbb{Z}(\forall y \in \mathbb{Z}\,((xy = 0) \iff (x = 0 \vee y = 0)))$

(b) Given some integer a, prove that $\forall x \in \mathbb{Z}((x-a)^2 = 0 \iff x = a)$

(c) Prove that $\forall x \in \mathbb{Z}(x^2 - 4x + 4 = 0 \iff x = 2)$

3. Suppose I am producing haute-couture fine art on Microsoft Paint, and I begin by opening a blank, white artwork and drawing two black lines. I extend the lines as far as possible so that they intersect the edge of the painting. I then use the color-fill tool, but due to time and dignity constraints, I am only allowed to use it three times. If my goal is to render as much of my artwork as possible into color, prove that there will be a white region left over if and only if the lines I drew intersect each other.

4. Suppose I have three logical statements a, b, c, and I want to prove they are all equivalent. That is, I wish to prove $a \iff b$, $b \iff c$, and $a \iff c$.

(a) Show that if $a \implies b$ and $b \implies c$ then $a \implies c$.

(b) Suppose we manage to prove $a \implies b$, $b \implies c$, and $c \implies a$. Is this enough to prove that $a \iff c$? [Hint: you can conclude $a \iff c$ from $a \implies c$ and $c \implies a$.]

(c) Is $a \implies b$, $b \implies c$, and $c \implies a$ enough to prove $a \iff b$ and $b \iff c$?

Proof by Contradiction

1. Recall that the product of any two positive real numbers is positive. Given real numbers x, y, z, such that $y > z$ and $xy < xz$, prove that $x < 0$.

2. Recall that an integer x is *even* if $x = 2k$ for some integer k, and x is *odd* if $x = 2k + 1$ for some integer k. Given two positive integers m, n such that mn is even, prove that m is even or n is even. (You may assume that any integer that is not even must be odd.)

3. Given that no positive integer is a factor of 1, prove the following statements:

(a) If x is even, then $x + 1$ is not divisible by 2.

(b) If x is divisible by 5, then $x + 1$ is not divisible by 5.

(c) If $x + 1$ is divisible by 5, then x is not divisible by 5.

(d) For any integer $x > 1$, x and $x + 1$ have no common factors. (A *common factor* is a positive integer k that divides both of the integers in question.)