MATH 8
LOGIC 4: LOGIC PROOFS
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Basic logic operations

$\neg$: NOT (for example, NOT A): true if A is false, and false if A is true.
$\wedge$: AND (for example A AND B): true if both A, B are true, and false otherwise.
$\vee$: OR (for example A OR B): true if at least one of A, B is true, and false otherwise.
$\implies$: IF... THEN (for example $A \implies B$: "if A, then B): if A is false, automatically true; if A is true, it is true only when B is true

Logic laws

We can combine logic operations, creating more complicated expressions such as $A \wedge (B \vee C)$. As in arithmetic, these operations satisfy some laws: for example $A \vee B$ is the same as $B \vee A$. Here, "the same" means "for all values of A, B, these two expressions give the same answer"; it is usually denoted by $\iff$. Here are two other laws:

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$
$$(A \implies B) \iff ((\neg B) \implies (\neg A))$$

Truth tables provide the easiest way to prove complicated logical rules: if we want to prove that two formulas are equivalent (i.e., always give the same answer), make a truth table for each of them, and if the tables coincide, they are equivalent.

Proofs

What is a proof?

Common answer is: a sequence of statements, startign with given ones and ending in a statement which we want to prove, and such that each statement in the sequence logically follows from the previous.

What exactly "logically follows" means?

In the simple case when all our statements can be written as combinations of the same elementary statements (which we can denote by letters $A, B, \dots$), using logical operations, it means the following:

> For any combinations of values of letters $A, B, \dots$ which makes the previous statements true, the next statement is also true.

Thus, it can be checked simply by a truth table. E.g., statement $\neg A$ logically follows from $A \implies B$ and $\neg B$: in all cases when $A \implies B$ and $\neg B$ are true, $\neg A$ is also true, as is easy to check.

However, usually instead of writing truth tables, people use some simple logical laws repeatedly. Here are some commonly used laws:

- Given $A \implies B$ and A, we can conclude B.

- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C.]
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$

These laws have some historical Latin names, such as *Modus Ponens*, but you do not need to know that :)

Laws used in Proofs

Here are some commonly used logic laws (all of them can be proved using truth tables):

- Given $A \implies B$ and A, we can conclude B (Modus Ponens)
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C.]
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ (De Morgan Law)
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ (law of contrapositive)

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

Common methods of proof

Conditional proof: To prove $A \implies B$, **assume** that A is true; derive B using this assumption.

Proof by contradiction: To prove A, assume that A is **false** and dervie a contradiction (i.e., something which is always false – e.g. $B \wedge \neg B$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

Homework

1. The following statement is sometimes written on highway trucks:

If you can't see my windows, I can't see you.

Can you write an equivalent statement without using word "not" (or its variations such as "can't").

2. Which of the statements below is equivalent to the statement "If you do not work hard, you fail the course":

(a) If you work hard, you do not fail the course.
(b) If you failed the course, you did not work hard.
(c) If you did not fail the course, you worked hard.

3. Consider the follwoing statement:

You can't be happy unless you have a clear conscience.

Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter H for "you are happy" and C for "you have a clear conscience".

Note: proving this statement is not part of the assignment :).

4. Use proof by contradiction to prove the following statement:

If the square of an integer number n is even, then n itself is even.

You can use without proof that every integer number is either even (i.e., can be written in the form $n = 2k$, with integre k) or odd (i.e., can be written as $n = 2k + 1$, with integer k).

5. Prove that the equation $x^3 + 7x + 19 = 0$ has no positive real roots. You can use all the usual properties of real numbers, in particular properties of the order relation $<$.

6. Prove by contradiction that there does not exist a smallest positive real number.

7. Prove the following logical equivalence:

$$\neg (p \implies q) \iff (p \wedge \neg q)$$

8. (a) Prove the following logical equivalence:

$$p \vee (q \vee r) \iff (p \vee q) \vee r$$

Thus the logical or $\vee$ is said to be *associative.*

(b) Prove that the logical and $\wedge$ is also associative.

(c) **Disprove** the following logical equivalence:

$$(p \implies (q \implies r)) \iff ((p \implies q) \implies r)$$

[Hint: which of the statements is true if p, q, and r are all false?] Conclude that the logical implication $\implies$ is not associative.

9. Given logical statements m, p, q, let a denote the combined statement $(m \wedge p) \vee (\neg m \wedge q)$. In other words, $a \iff ((m \wedge p) \vee (\neg m \wedge q))$. Prove the following:

(a) If m is true, then $a \iff p$

(b) If m is false, then $a \iff q$

10. Asha, Benedict, and Cerys have bought a cake, but alas, due to bad planning, they are not sure if they all actually like the cake. To solve this issue in a terribly inefficient way, they decide to play a game: they find three lights, which we will call 1, 2, and 3, and if Asha likes the cake, Asha will switch lights 1 and 2; if Benedict likes the cake, Benedict will switch lights 2 and 3; if Cerys likes the cake, Cerys will switch lights 1 and 3. Initially all the lights are off (note that if a light that is off is switched, it turns on, and if a light that is on is switched, it turns off). Let us denote three logical statements: a will mean Asha likes the cake, b will mean Benedict likes the cake, and c will mean Cerys likes the cake. Thus, for example, if a, b, and c are all false, then all the lights remain off.

(a) If $a \iff b \iff c$, then which lights are on?

(b) If $(a \iff \neg b) \wedge (a \iff \neg c)$, then which lights are on?

(c) Now let l_1, l_2, and l_3 denote the logical statements that lights 1, 2, 3 are switched on, respectively. If $\neg((l_1 \wedge l_2) \implies l_3)$ and exactly one person likes the cake, then who likes the cake?

(d) If $\neg(l_1 \implies l_3) \vee \neg(l_2 \implies l_3)$ and exactly two people like the cake, then who likes the cake?