

MATH 7: HANDOUT 18

COORDINATE GEOMETRY 3: PARABOLAS

REVIEW OF QUADRATIC EQUATIONS

Here is what we have learned so far about quadratic equations:

- A **quadratic polynomial** is an expression of the form $p(x) = ax^2 + bx + c$.
- **Roots** of a quadratic polynomial are numbers such that $p(x) = 0$. If x_1, x_2 are roots, then $p(x) = a(x - x_1)(x - x_2)$.
- **Vietá formulas:** If x_1, x_2 are roots of $x^2 + bx + c$, then

$$\begin{aligned}x_1 + x_2 &= -b \\x_1 x_2 &= c\end{aligned}$$

- **Completing the square:** we can rewrite

$$(1) \quad ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a} = a \left(\left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right)$$

where $D = b^2 - 4ac$.

From this, one gets the **quadratic formula**: if $D < 0$, there are no roots; if $D \geq 0$, then the roots are

$$(2) \quad x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

- From formula (1), we see that:
 - If $a > 0$, then the **smallest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going up.
 - If $a < 0$, then the **largest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going down.

GRAPHS OF QUADRATIC FUNCTIONS

- We know how to draw the graph of $y = x^2$. It's a parabola.
- We know that the graph of $y = x^2 + b$ can be obtained from the graph of $y = x^2$ by shifting up by b units (or down, if $b < 0$)
- We know that the graph of $y = (x + a)^2$ can be obtained from the graph of $y = x^2$ by shifting left by a units (or right, if $a < 0$).
- Based on the two facts above, we can draw a graph of any function of the type $y = (x + a)^2 + b$.

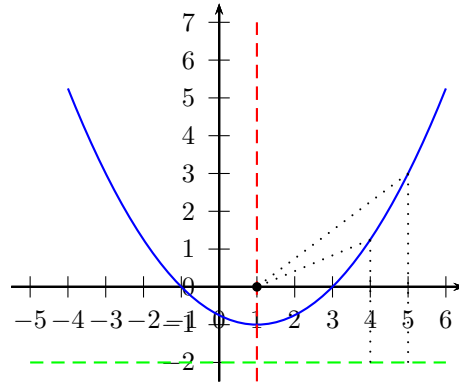
We can transform any quadratic function $y = x^2 + px + q$ to $y = (x + a)^2 + b$ by completing the square.

PROPERTIES OF A PARABOLA

A parabola is the set of all points in a plane that are equally distant away from a given point and a given line (see black dotted lines).

This given point is called the **focus** (black dot) of the parabola and the line is called the **directrix** (green line).

If the parabola is of the form $(x - h)^2 = 4p(y - k)$, the vertex is (h, k) , the focus is $(h, k + p)$ and directrix is $y = k - p$



HOMEWORK

1. For what values of a does the polynomial $x^2 + ax + 14$ has no roots? exactly one root? two roots?
2. Let x_1, x_2 be the roots of the equation $x^2 + 3x + 4 = 0$. Without calculating the roots, find:
 - (a) $x_1^2 + x_2^2$
 - (b) $\frac{1}{x_1^2} + \frac{1}{x_2^2}$
3. A circle with center $(3, 5)$ intersects the y -axis at $(0, 1)$.
 - (a) Find the radius of the circle
 - (b) Find the coordinates of the other point of intersection on the y -axis
 - (c) What are the coordinates of the intersection points of the circle with the x -axis?
4. Of all the rectangles with perimeter 4, which one has the largest area?
 [Hint: if sides of the rectangle are a and b , then the area is $A = ab$, and the perimeter is $2a + 2b = 4$. Thus, $b = 2 - a$, so one can write A using only a . . .]
5. Prove that for any point P on the parabola $y = \frac{x^2}{4} + 1$, the distance from P to the x -axis is equal to the distance from P to the point $(0, 2)$.
6. Use completing the square method to draw the following graphs:

(a) $y = x^2 - 5x + 5$	(d) $y = -x^2 + 3x - 0.5$
(b) $y = x^2 - 4x + 2$	(e) $y = x^2 + 4x - 4$
(c) $y = x^2 - x - 1$	
7. Graph $y = (\sqrt{x})^2$. Note $x \geq 0$
8. A triangle ABC has corners $A(-3, 0)$, $B(0, 3)$ and $(3, 0)$. The line $y = \frac{1}{3}x + 1$ separates the triangle in 2. What is the area of the piece lying below the line?