MATH 7: HANDOUT 16
MORE INEQUALITIES. SNAKE METHOD.

Solving polynomial inequalities

In addition to linear inequalities, we can also consider polynomial inequalities: they would have terms like x^2, x^3, etc. The general rule for solving polynomial inequalities is as follows:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- Roots $x_1, x_2, \dots$ divide the real line into intervals; define the sign of each factor and the product on each of the sign intervals.
- If the inequality has $\geq$ or $\leq$ signs you should also include the roots themselves into the intervals.

Example 1. $x^2 + x - 2 > 0$.

Solution. We find roots of the equation $x^2+x-2 = 0$ and obtain $x = -2, 1$. The inequality becomes $(x+2)(x-1) > 0$ and roots -2, 1 divide the real line into three intervals $(-\infty, -2), (-2, 1), (1, +\infty)$. It is easy to see that the polynomial $x^2 + x - 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then $x < -2$ or $x > 1$. We sometimes, write this also as $x \in (-\infty, -2) \cup (1, +\infty)$. (sign $\cup$ means "or").

Example 2. $-x^2 - x + 2 \geq 0$.

Solution. We have $-(x + 2)(x - 1) \geq 0$. The left hand side is positive for $-2 < x < 1$. As the sign in the inequality is $\geq$ we have to includes the roots into the interval and obtain $-2 \leq x \leq 1$. One can also write $x \in [-2, 1]$ (square brackets here mean that the endpoints of the interval are included).

Example 3. $x^2 + x + 2 \geq 0$.

Solution. The polynomial here does not have roots (the discriminant $12 - 4 \cdot 1 \cdot 2 < 0$). Therefore, the real line is not divided into the intervals, which means that the polynomial is of the same sign for all x. We check that it is positive, for example, for $x = 0$. The solution is that x is any number. We can write $x \in (-\infty, +\infty)$.

Example 4. $x^2 + x + 2 < 0$.

Solution. The polynomial does not have roots and is positive everywhere. This means that the inequality does not have solutions at all. One can also write $x \in \emptyset$.

Example 5. $x^2 - 2x + 1 > 0$.

Solution. The inequality is $(x - 1)^2 > 0$. There is only one root here which divides the real line into two intervals. The solution is $x < 1$ or $x > 1$, that is any x except for $x = 1$. One can write $x \in (-\infty, 1) \cup (1, +\infty)$.

Same method can be used to solve any polynomial inequality, for example $x^n + bx^{n-1} + \cdots \geq 0$, where n is greater than 2 — but we need to know the way to either find the roots of the corresponding equation or to have factorization given to us.

Example 6. Solve the inequality $(x + 1)(x - 2)^2(x - 4)^3 \geq 0$.

Solution. Notice that if we solve the corresponding equation $(x + 1)(x - 2)^2(x - 4)^3 = 0$, we get $x = -1, 2, 4$. Therefore, we need to consider the following 4 intervals: $(-\infty; -1), (-1; 2), (2; 4), (4; \infty)$.

Notice that in the 1st interval, the expression $(x + 1)(x - 2)^2(x - 4)^3$ is positive, and therefore satisfies the inequality.

Then, as x "crosses" point 1, the expression changes its sign to '$-$', and therefore the interval $(-1; 2)$ does not satisfy the inequality.

Now, crossing point 2 again won't change the sign of the expression, because $(x-2)^2$ is always positive. Therefore, the interval $(2; 4)$ also doesn't satisfy the inequality.

Finally, crossing point 4, the expression changes its sign to '$+$', and therefore the interval $(4; \infty)$ satisfies the inequality. So, the answer to the inequality is:

$$x \in (-\infty; -1] \cup 2 \cup [4; \infty)$$

The method used to solve this problem is called a **snake method**.

Example 7. Solve the inequality $\frac{(x+1)(x-2)^2}{(x-4)^3} \geq 0$.

Solution. Note that the factors in this inequality are exactly the same as in the previous example, so the solution will be the same with the small (but important) exception: the denominator cannot be equal to 0, and therefore, x cannot be equal to 4 — notice the round instead of square bracket in the answer!

$$x \in (-\infty; -1] \cup 2 \cup (4; \infty)$$

Inequalities with Absolute Value

When you have an inequality with absolute value, you will have to consider various cases: when the expression under absolute value is positive and when the expression under the absolute value is negative, and use the definition of the absolute value:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } -x \geq 0 \end{cases}$$

Example 8. Solve inequality $|x-4| < 7$.

Solution. Solution: Again, as before, we need to consider two cases, the one when $x - 4 \geq 0$ and the one when $x - 4 < 0$.

Case 1. $x - 4 \geq 0$ means that $x \geq 4$. Now, since $x - 4 \geq 0$, we have $|x-4| = x - 4$, and the inequality can be rewritten as

$$x - 4 < 7$$

Solving this inequality gives us $x < 11$. But remember, x must be greater than or equal to 4! So, combining both things together, we get $4 \leq x < 11$, or $x \in [4; 11)$.

Case 2. $x - 4 < 0$ means that $x < 4$. Now, since $x - 4 < 0$, we have $|x - 4| = -(x - 4) = 4 - x$, and the inequality can be rewritten as

$$4 - x < 7$$

Solving this inequality gives us $x > -3$. But remember, x must also be less than 4! So, combining both things together, we get $-3 < x \leq 4$.

Combining Cases 1 and 2 together, we get the final solution to the inequality: $-3 < x < 11$ or

$$x \in (-3, 11)$$

HOMEWORK

1. Solve the following equations.

(a) $|x-3|=5$

(b) $|2x-1|=7$

(c) $|x^2-5|=4$

2. Solve the following equations.

(a) $\dfrac{(x+1)}{(x-1)}=3$

(b) $\dfrac{(x^2-9)}{(x+1)}=(x+3)$

(c) $x-\dfrac{3}{x}=\dfrac{5}{x}-x$

3. Solve the following inequalities, show solution on the real line, write the answer in the interval notation.

(a) $|x-2|>3$

(b) $|x-1|>x+3$

(c) $\dfrac{(x-2)}{(x+3)}\leq 3$

4. Solve the following quadratic equations and inequalities:

(a) $x^2+2x-3=0, \qquad x^2+2x-3>0, \qquad x^2+2x-3\leq 0$

(b) $x^2+2x+3=0, \qquad x^2+2x+3\geq 0, \qquad x^2+2x+3<0$

(c) $-x^2+6x-9=0, \qquad -x^2+6x-9\geq 0, \qquad -x^2+6x-9<0$

(d) $3x^2+x-1=0, \qquad 3x^2+x-1\geq 0, \qquad 3x^2+x-1\leq 0$

5. Solve the following inequalities, using the snake method. Show solution on the real line. Write the answer in the interval notation.

(a) $(x-1)(x+2)>0$

(b) $(x+3)(x-2)^2\leq 0$

(c) $x(x-1)(x+2)\geq 0$

(d) $x^2(x+1)^5(x+2)^3>0$

***6.** Solve the following inequalities, using the snake method. Show solution on the real line. Write the answer in the interval notation.

(a) $|x^2-x|\geq 2x$

(b) $\dfrac{x(x-1)^2}{(x+1)^2}\geq 0$