MATH 7: HANDOUT 10
POKER PROBABILITIES

Poker Probabilities

In the game of poker, a player is dealt five cards from a regular deck with 4 suits (♠, ♣, ♢, ♡) with card values in the following order: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. We calculated probabilities of the following combinations:

Royal Flush: 10, J, Q, K, A of any suit (Example: 10♡, J♡, Q♡, K♡, A♡)
There are only 4 of them.

Straight Flush: Five cards in a row of the same suit (Example: 6♡, 7♡, 8♡, 9♡, 10♡)
Each of these can start from any card from A to 9, and be in each of the for suits: $9 \times 4 = 36$. Notice that we excluded royal flushes from out computation (if we start with 10, we get a Royal Flush).

Four of a kind: Four cards of the same value, and one other random card (Example: K♡, K♠, K♢, K♣, 2♣)
Which card $13\times$ Which other value $12\times$ Which other suit $4 = 13 \cdot 12 \cdot 4$.

Full House: Three cards of the same value, and two cards of the same value (Example: K♡, K♠, K♢, 4♠, 4♣)
Which card for 3 $13\times$ Which three suits $\binom{4}{3}\times$ Which card for a pair $12\times$ Which two suits $\binom{4}{2} = 13\binom{4}{3}\cdot 12\binom{4}{2}$.

Flush: Five cards of the same suit, not in order (Example: 3♡, 6♡, 8♡, J♡, A♡)
Which suit $4\times$ Which five cards $\binom{13}{5} = 4\binom{13}{5}$. We also need to exclude Royal Flushes and Straight Flushes, so the total is $4\binom{13}{5} - 40$.

Straight: Five cards in order, possibly of different suits (Example: 5♡, 6♠, 7♢, 8♠, 9♣)
Which card to start from (anything from A to 10) $10\times$ Five suits $4^5 = 10 \cdot 4^5$. From here we also need to exclude Royal Flushes and Straight Flushes, so the final answer if $10 \cdot 4^5 - 40$.

Triple: Three cards of the same value, and two other random cards (Example: K♡, K♠, K♢, 4♠, 2♣)
Which card $\binom{13}{1}\times$ Which three suits $\binom{4}{3}\times$ Which two other values $\binom{12}{2}\times$ Which two suits for these two random card $4^2 = \binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$.

Two pairs: Two cards of the same value, two cards of the same value, and a random card (Example: K♡, K♠, 10♢, 10♠, 4♣)
Which two cards $\binom{13}{2}\times$ Two suits for each of pair $\binom{4}{2}^2\times$ Remaining value $11\times$ Remaining suit $4 = \binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$.

Pair: Two cards of the same value, and three other random cards (Example: K♡, K♠, Q♢, 4♠, 2♣)
Which card $\binom{13}{1}\times$ Which two suits $\binom{4}{2}\times$ Which three other values $\binom{12}{3}\times$ Which three suits for these three random card $4^3 = \binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$.

To calculate probabilities of each of these combinations, we have to divide the counts above by the total number of poker hands, which is $\binom{52}{5}$. The table below gives the probabilities and odds:

Combination	Count	Probability	Odds
Royal Flush	4	0.000154%	1 : 649,740
Straight Flush	36	0.00139%	1 : 72,192
Four of a Kind	$13 \cdot 12 \cdot 4$	0.024%	1 : 4,165
Full House	$13\binom{4}{3} \cdot 12\binom{4}{2}$	0.1441%	1 : 693
Flush	$4\binom{13}{5} - 40$	0.1965%	1 : 508
Straight	$10 \cdot 4^5 - 40$	0.3925%	1 : 254
Triple	$\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$	2.1128%	1 : 46.3
Two Pairs	$\binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$	4.7539%	1 : 20
Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$	42.2569%	1 : 1.37
Nothing		50.1177%	1 : 0.995

Homework

1. (a) How many 10-letter "words" one can write using 4 letters H and 6 letters T?
   (b) If we toss a coin 10 times and record the result as a sequence of letters H and T (writing H for heads and T for tails), how many different possible sequences we can get? How many of them will have exactly 6 tails?
   (c) If we toss a coin 10 times, what are the chances that there will be 6 tails? 3 tails? at least one tails?

2. If we randomly select 100 people form the population of the US, what are the chances that exactly 50 of them will be males? that at least 50 will be males? that all 100 will be males?

3. How many ways are there to divide 12 books
   (a) Between two bags
   (b) Between two bookshelves (order on each bookshelf matters!)
   (c) Between three bags
   (d) Between three bookshelves (order on each bookshelf matters!)

4. A person is running down the staircase. He is in a rush, so he may jump over some steps. If the staircase is 12 steps (including the top one, where he begins, and the last one, where he ends), in how many ways can he reach the bottom step in 5 jumps? What if there are no restrictions on the number of jumps? [Hint: keep track of the steps he steps on. . .]

5. [We talked about this problem before... Let's do it again!]
   (a) For a group of 25 people, we ask each of them to choose a day of the year (non-leap, so there are 365 possible days). How many possible combinations can we get? [Order matters: it is important who had chosen which date]
   (b) The same question, but now we additionally require that all chosen dates be different.
   (c) In a group of 25 people, what are the chances that no two of them have birthday on the same day? that at least two have the same birthday?