MATH 7: HANDOUT 9
BINOMIAL PROBABILITIES

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Binomial coefficients are also useful in calculating probabilities.
Let us first introduce the terminology:

Trial: one instance of an experiment. For example, if we are doing a sequence of coin tosses, each coin toss is a *trial*. If we are shooting ducks, each shot is a *trial*.

n**:** Number of trials is denoted by n.

Success: a trial that ends up in a desired outcome. If we are looking for Heads, *success* is an outcome of getting a Head. If we are looking at duck shooting, *success* is a hit.

k**:** Number of successes we need.

p**:** Probability of success in one trial ($1/2$ for heads/tails)

Failure: a trial that does not end up in a success (missing a duck, getting a Tail while looking for Heads)

q**:** Probability of failure, $q = 1 - p$.

Imagine that we want to calculate the probability of getting k successes in n trials, and we know p. Then:

$$P(k \text{ successes in } n \text{ trials }) = \binom{n}{k} p^k q^{n-k}, \text{ where}$$

- p — probability of success in one try;
- $q = 1 - p$ — probability of failure in one try;
- n — number of trials;
- k — number of successes;
- $n - k$ — number of failures.

Example: You roll a 6-sided die 6 times. What is the probability of getting a 6 exactly once?
Solution: Here we have: $n = 6$, $k = 1$, $p = 1/6$, $q = 5/6$. Then

$$P = \binom{6}{1} \cdot \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = 6 \cdot \frac{5^5}{6^5} = \frac{5^5}{6^6}.$$

Example: A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$. What is the probability that out of 7 shots, she will hit exactly three times?
Solution: Here we have: $n = 7$, $k = 3$, $p = 1/3$, $q = 2/3$. Then

$$P = \binom{7}{3} \cdot \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{7-3} = \binom{7}{3} \cdot \frac{2^4}{3^7}.$$

HOMEWORK

In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use Pascal triangle: $\binom{n}{k}$ is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. A (blindfolded) marksman finds that on the average he hits the target 4 times out of 5. If he fires 4 shots, what is the probability of
   (a) more than 2 hits?
   (b) at least 3 misses?

2. In each of 4 races, the Democrats have a 60% chance of winning. Assuming that the races are independent of each other, what is the probability that:
   (a) The Democrats will win 0 races, 1 race, 2 races, 3 races, or all 4 races?
   (b) The Democrats will win at least 1 race.
   (c) The Democrats will win a majority of the races.

3. The ratio of boys to girls at birth in Singapore is quite high at $1.09 : 1$.
   What proportion of Singapore families with exactly 6 children will have at least 3 boys?

4. [Same as Problem 4 in Handout 8] A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$.
   (a) The hunter makes 5 shots. What is the probability that she misses all five?
   (b) What is the probability that out of 5 shots, she will hit a duck at least once? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)
   (c) What is the probability that out of 5 shots, she will hit exactly once? Will this probability double is she makes 10 shots?
   (d) What is the probability that out of 5 shots, she will hit a duck exactly three times? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)
   (e) What is the probability that she hits a duck half times or more if she fires 5 times (that is, 3, 4, or 5 hits)? What about if she fires 10 times (that is 5, 6, 7, 8, 9, or 10 hits)?
   (f) What is the most likely number of hits out of 5 shots? And out of 10 shots?

5. [Same as Problem 5 in Handout 8] At a fair, they offer you to play the following game: you are tossing small balls in a large crate full of empty bottles; if at least one of the balls lands inside a bottle, you win a stuffed toy (worth about $5). Unfortunately, it is really impossible to aim, so the game is just a matter of luck (or probability theory): every ball you toss has a 20% probability of landing inside the bottle.
   (a) If you are given three balls, what is the probability that all three will be hits? That all three will be misses? That at least one will be a hit?
   (b) Same questions for five balls.
   (c) What about seven balls?
   (d) How much should the organizers charge for 3 balls to break even? What about for 5 balls?