MATH 7: HANDOUT 7
PASCAL TRIANGLE AND ITS APPLICATIONS

Pascal triangle

Recall the Pascal triangle:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Every entry in this triangle is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by $\binom{n}{k}$. Note that both n and k are counted from 0, not from 1: for example, $\binom{6}{2} = 15$.

In the previous handout we saw that these numbers appear in a problem about counting paths from the lower left corner of the board to the upper right corner. We observed the following:

$$\binom{n}{k} = \text{the number of paths on the chessboard going } k \text{ units up and } n-k \text{ units to the right}$$

For example, the number of paths that go to the upper right corner of a 6×6 board is equal to $\binom{10}{5}$, as each such path must have 5 steps to the right and 5 steps up. Now let us think about other applications of these numbers.

Words with 1s and 0s: Each on the board going up and to the right can be written as a sequence of steps, R for the step to the right, an U as a step up. For example, a path RRRRRUUUUU will go five step to the right and five steps up, eventually ending in the upper right corner of a 6×6 board. There is a correspondence between paths of length n and strings of length n consisting of Rs and Us only. Now let us switch Rs to 0s and Us to 1s. Now, we already know that $\binom{n}{k}$ is a number of paths going k units up and $n-k$ units to the right, which corresponds to words of length n with k Us and $n-k$ Rs, which is the same as a number of strings of length n with k 1s and $n-k$ Us. We have the following result:

$$\binom{n}{k} = \text{the number of words that can be written using } k \text{ ones and } n-k \text{ zeroes}$$

Combinations: Now, let us consider all words on length n with k ones. The number of such words in $\binom{n}{k}$, as we showed above. Consider now a set on n elements, and let's number them from 1 to n. Now for each string on length n with 0s and 1s, we can select those elements that corresponds to 1s and omit those elements that correspond to 0s. That way, we will get a subset of size k. This way, we get another property of binomial coefficients:

$$\binom{n}{k} = \text{the number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}$$

To summarize, this is what we got:

$$\binom{n}{k} = \text{the number of paths on the chessboard going } k \text{ units up and } n-k \text{ to the right}$$
$$= \text{rhe number of words that can be written using } k \text{ ones and } n-k \text{ zeroes}$$
$$= \text{the number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}$$

Homework

In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use Pascal triangle: $\binom{n}{k}$ is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. How many “words” of length 5 one can write using only letters U and R, namely 3 U's and 2 R's? What if you have 5 U's and 3 R's? [Hint: each such “word” can describe a path on the chessboard, U for up and R for right. . .]

2. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?

3. If we toss a coin 10 times, what is the probability that all will will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?

4. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub — and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will move
   (a) 10 steps north
   (b) 10 steps south
   (c) 4 steps north
   (d) will come back to the starting position

5. If you have a group of 4 people, and you need to choose one one to go to a competition, how many ways are there to do it? if you need to choose 2? if you need to choose 3?

6. How many ways are there to select 5 students from a group of 12?

7. In a meeting of 25 people, each much shake hands with each other. How many handshakes are there altogether?

8. (a) An artist has 12 paintings. He needs to choose 4 paintings to include in an art show. How many ways are there of doing this?
   (b) The same artist now needs to choose 4 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).