Homework 15: Vieta's formulas

HW15 is Due January 31; submit to Google classroom 15 minutes before the class time.

1. Solving the complete quadratic equation

- By completing the square

"Completing the square" works by using the formulas for fast multiplication $(a \pm b)^2 = a^2 \pm 2ab + b^2$ (*)
Here is an example how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x+3)^2 - 7 = (x+3)^2 - \left(\sqrt{7}\right)^2 = \left(x + 3 + \sqrt{7}\right)\left(x + 3 - \sqrt{7}\right)$$

Thus, $x^2 + 6x + 2 = 0$ if and only if $(x + 3 + \sqrt{7}) = 0$, which gives $x = -3 - \sqrt{7}$, or if $(x + 3 - \sqrt{7}) = 0$, which gives $x = -3 + \sqrt{7}$.

- By using the quadratic formula

Steps: for the equation in the standard form $ax^2 + bx + c = 0$

List coefficients: $a =, b =, c =$
Find the determinant D: $\boldsymbol{D = b^2 - 4ac}$
Check the number of roots (solutions): The determinant, D, determines the number of solutions.
If D < 0, there are no real solutions; if D = 0, there is one solution, if D > 0, there are two solutions.
Find the solutions: $\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}}$

- Vieta's formulas

Of course, with the quadratic formula we can always solve any quadratic equation, and then do the operations with its roots. However, sometimes it is unnecessary. The formulas below are called Vieta's Formulas and allow us to find the sum and the product of roots of a quadratic equation without explicitly calculating them. If you would like to use the formulas to find (guess) the roots, this will be easier if the coefficients a, b, c are whole numbers.

for equation in standard form: $ax^2 + bx + c = 0$

The roots of the quadratic equation are related to the coefficients: $\boldsymbol{x_1 x_2 = \frac{c}{a}}$ and $\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$
In the special case when $\boldsymbol{a = 1}$, $\boldsymbol{x_1 x_2 = c}$ and $\boldsymbol{x_1 + x_2 = -b}$

How did we get to the Vieta's formulas? if x_1 and x_2 are the roots, then the quadratic equation can be written from standard into factored form:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

For the special case when a = 1, we set a = 1 and then multiplying the expressions in the right hand side, we get: $x^2 + bx + c = (x - x_1)(x - x_2) = x^2 - x_1 - x_2 x + x_1 x_2 = x^2 - (x_1 + x_2)x + \boldsymbol{x_1 x_2}$

From comparing the numbers in front of the same powers of x the coefficients we can get the following:

$$x_1 + x_2 = -b$$
$$x_1 x_2 = c$$

2. **formulas for fast multiplication** $a^2 - b^2 = (a-b)(a+b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Find the roots of the equations x_1 and x_2 using Vieta's formulas. After that, write the equation in a factorized form as $a(x - x_1)(x - x_2)$.
   a. $x^2 - 5x + 6 = 0$
   b. $x^2 + 8x - 9 = 0$
   c. $2x^2 + 4x - 6 = 0$

2. (See our class problem 5) Let x and y be some numbers. Use the formulas for fast multiplication to rewrite the following expressions using only $(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{B}$ and $\boldsymbol{xy} = \boldsymbol{C}$, where B and C are just number. To do that, present the expressions as sums and/or products of x and y, then substitute the sums/products with B and C. No x and y are allowed in the answers!

Example: (*5d.* from the class problems): $x^2 + y^2 = x^2 + y^2 + 2xy - 2xy = (x + y)^2 - 2xy = B^2 - 2C$
We completed the square using the formula: $a^2 \pm 2ab + b^2 = (a \pm b)^2$

   a. $(x - y)^2 =$
   b. $\frac{1}{x} + \frac{1}{y} =$
   c. $\frac{1}{x-1} + \frac{1}{y-1} =$
   d. $x^2 - y^2 =$
   e. $x - y =$
   Hint: use $x^2 - y^2 = (x + y)(\boldsymbol{x} - \boldsymbol{y})$
   f. (*) $x^3 + y^3$
   Hint: first compute $(x + y)(x^2 + y^2)$

3. Let x_1, x_2 be the roots of the equation $x^2 + 5x - 7 = 0$. Using the Vieta's formulas, find the values of the expressions *without* explicitly calculating x_1 and x_2
   a. ${x_1}^2 + {x_2}^2 =$
   b. $(x_1 - x_2)^2 =$
   c. $\frac{1}{x_1} + \frac{1}{x_2} =$
   d. (*) ${x_1}^3 + {x_2}^3 =$

4. Solve the following equations.
   a. $x^2 - 5x + 6 = 0$
   b. $5x^2 = 8x - 3$
   c. $\sqrt{2x + 1} = x$
   d. $x + \frac{2}{x} = 3$

5. Solve the biquadratic equation: $x^4 - 3x^2 + 2 = 0$

6. Prove the following inequalities:
   a. for any $a > 0$, prove that $a + \frac{1}{a} \geq 2$, with equality only when a = 1.
   b. Show that for any $a, b \geq 0$, one has

$$\frac{a + b}{2} \geq \sqrt{ab}.$$

The left hand side is usually called the arithmetic mean of a, b; the right hand side is called the geometric mean of a, b .

Hint: I suggest you "remove" the denominator or the radicals (square roots), write the quadratic equations into factorized form, with zero on one side of the new inequality. Try your best, we will be solving inequalities soon.