MATH 6
HANDOUT 29: FIBONACCI NUMBERS

Definition 1. The Fibonacci numbers is the sequence of numbers constructed by the following rule: $F_1 = F_2 = 1$ and for $n > 1$, $F_{n+1} = F_n + F_{n-1}$. Here are the first several Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, ...

(Some people also define $F_0 = 0$.)

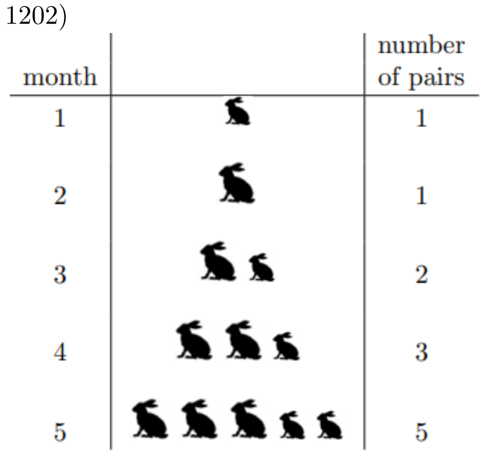
Homework

1. Find the sum $1 + 11 + 111 + \cdots + 11\ldots11$, where the last number contains 57 ones. [Hint: find the sum $9 + 99 + 999 + \cdots + 99\ldots99$ and divide by 9.]

2. On Halloween, Mark has collected 779 pieces of candy. If he starts eating them on November 1st, eating one piece on the first day, two pieces on the second day, three pieces on the third day and so on, how long will his candy last?

3. (attributed to Leonardo of Pisa, also called Fibonacci, 1202)

Somebody buys a pair of rabbits and places them in a pen. The nature of rabbits is such that each month a pair of rabbits gives birth to another pair, and they start reproducing upon reaching the age of 2 months.
How many pairs of rabbits will he have in one year (considering the rabbits immortal)?



[attributed to certain Leonardo of Piza, also called Fibonacci, 1202]

4. Show that for any n, $F_1 + F_2 + \ldots F_n = F_{n+2} - 1$. [Hint: $F_{n+2} = F_{n+1} + F_n$.]

5. (a) Which of Fibonacci numbers are even? Find the pattern and try to explain why this patterns holds.
(b) Which of Fibonacci numbers are divisible by 3? Find the pattern and try to explain why this patterns holds.

Definition 2. A generalized Fibonacci sequence (GFS) is any sequence $x_1, x_2, \ldots$ such that $x_{n+1} = x_n + x_{n-1}$ (x_1, x_2 can be any numbers). For example, $1, 3, 4, 7, \ldots$ is a GFS.

The geometric progressions $1, \Phi, \Phi^2, \ldots$ and $1, \bar{\Phi}, \bar{\Phi}^2, \ldots$ where $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$, $\bar{\Phi} = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots$, are GFS.

The number Φ appears in many places in mathematics. It is called the *Golden Ratio* (in old times, it was also sometimes called the *Divine proportion*). There are whole books about it; the problems below is just one of the illustrations of how this number appears. By the way, Φ is the Greek letter which reads "phi".

6. Consider the rectangle with sides 1 and Φ. Show that if we cut from it a 1×1 square, then the remaining rectangle will again have proportions $1 : \Phi$.