MATH 6: THE TRIANGLE

1. Sum of the Angles in a Triangle Add Up to 180^0

In any given triangle, its angles add up to 180^0.
For a triangle, any exterior angle is equal to the sum of the two interior non-adjacent angles.

2. Congruence

In general, two figures are called congruent if the have same shape and size. We use symbol $\cong$ for denoting congruent figures: to say that M_1 is congruent to M_2, we write $M_1 \cong M_2$.
Precise definition of what "same shape and size" means depends on the figure:

- For line segments, it means that they have the same length: $\overline{AB} \cong \overline{CD}$ is the same as $AB = CD$.
- For angles, it means that they have the same measure: $\angle A \cong \angle B$ is the same as $m\angle A = m\angle B$.
- For triangles, it means that the corresponding sides are equal and corresponding angles are equal: $\triangle ABC \cong \triangle A'B'C'$ is the same as
  $AB = A'B'$, $BC = B'C'$, $AC = A'C'$, $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$.

Note that for triangles, the notation $\triangle ABC \cong \triangle A'B'C'$ not only tells that these two triangles are congruent, but also shows which vertex of the first triangle corresponds to which vertex of the second one. For example, $\triangle ABC \cong \triangle PQR$ is not the same as $\triangle ABC \cong \triangle QPR$.

3. Congruence tests for triangles

By definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalitites. However, it turns out that in fact, we can do with fewer checks.

Angle-Side-Angle Congruence Axiom (ASA) If $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $AB = A'B'$, then $\triangle ABC \cong \triangle A'B'C'$.

Side-Side-Side Congruence Axiom (SSS) If $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.

Side-Angle-Side Congruence Axiom (SAS) If $AB = A'B'$, $AC = A'C'$ and $m\angle A = m\angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.

4. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

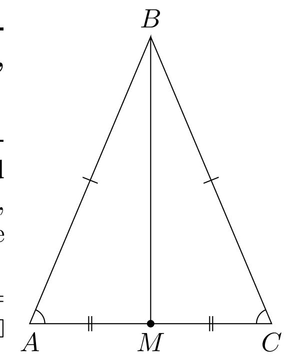
Base angles are equal If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

Proof. Assume that $\triangle ABC$ is isoceles, with apex B. Then by SAS, we have $\triangle ABC \cong \triangle CBA$. Therefore, $m\angle A = m\angle C$. Can you prove the converse? □

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide (for the apex).

If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.



Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and $m\angle MAB = m\angle MCB$ (by Base Angles Theorem). Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

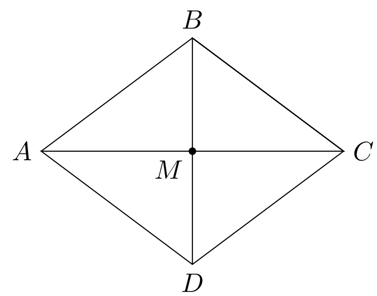
Homework

1. Do problem 6 on page 120, and problems 16,17,18,19,20 on page 146 in the textbook.

2. Let $\triangle ABC$ be such that all sides have equal length. Prove that then $m\angle A = m\angle B = m\angle C = 60^\circ$. [Such a triangle is called equilateral.]

3. Prove that if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

4. Let $ABCD$ be a quadrilateral such that $AB = BC = CD = AD$ (such a quadilateral is called rhombus). Let M be the intersection point of AC and BD.
   (a) Show that $\triangle ABC \cong \triangle ADC$
   (b) Show that $\triangle AMB \cong \triangle AMD$
   (c) Show that the diagonals are perpendicular and that the point M is the midpoint of each of the diagonals.



5. The following method explains how one can find the midpoint of a segment AB using a ruler and compass:
- Choose radius r (it should be large enough) and draw circles of radius r with centers at A and B.
- Denote the intersection points of these circles by P and Q. Draw a line $\overleftrightarrow{PQ}$.
- Let M be the intersection point of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. Then M is the midpoint of AB.

Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of AB? You can use the defining property of the circle: for a circle of radius r, the distance from any point on this circle to the center is exactly r.[Hint: use the previous problem]