MATH 6: HANDOUT 12

1. Geometric Sequences

A sequence of numbers is a geometric sequence if the next number in the sequence is the current number times a constant, called the **common ratio**. For example, let's consider the sequence: 6, 12, 24, 48, ... The first term in the sequence is 6, the second is $6 \cdot 2 = 12$, the third is $12 \cdot 2 = 24$ and so on. The common ratio is 2. What is the 10^{th} term? What is the n-th term?

$b_1 = 6$
$b_2 = 6 \cdot 2 = 12$
$b_3 = (6 \cdot 2) \cdot 2 = 6 \cdot 2^2 = 24$
$b_4 = (6 \cdot 2^2) \cdot 2 = 6 \cdot 2^3 = 48$
...
$b_{10} = (6 \cdot 2^9) \cdot 2 = 6 \cdot 2^9 = 3072$
In general, given the common difference q, the n^{th} term, $b_n = b_1 \cdot q^{n-1}$.

2. Geometric Mean

A property of a geometric sequence is that any term is the geometric mean of its neighbors. For example, $b_3 = \sqrt{b_2 \cdot b_4} = \sqrt{12 \cdot 48} = \sqrt{12 \cdot 12 \cdot 4} = 12 \cdot 2 = 24$. In general,

$$b_n = \sqrt{b_{n-1} \cdot b_{n+1}}$$

3. Sum of a Geometric Sequence

Let's find the sum for the powers of 2, $S = 1 + 2 + 4 + ... + 64$. The common ratio is 2. Multiply S with the common ratio (1b), then subtract S (1c).

(1a) $$S = 1 + 2 + 4 + ... + 64$$

(1b) $$2 \cdot S = 2 + 4 + ... + 64 + 128$$

(1c) $$2 \cdot S - S = 2 + 4 + .. + 64 + 128 - (1 + 2 + 4 - ... + 64) = 128 - 1 = 127$$

Let's consider the general case. Let b_1, ... b_n be a geometric sequence with common ratio q. Then,

(2a) $$S = b_1 + b_2 + ... + b_n$$

(2b) $$q \cdot S = q \cdot b_1 + q \cdot b_2 + ... + q \cdot b_{n-1} + q \cdot b_n = b_2 + b_3 + ... + b_n + b_{n+1}$$

(2c) $$q \cdot S - S = b_2 + b_3 + ... + b_n + b_{n+1} - (b_1 + b_2 + ... + b_n) = b_{n+1} - b_1 = b_1 \cdot q^n - b_1$$

(2d) $$S = b_1 \cdot \frac{(q^n - 1)}{(q - 1)}$$

4. Homework

1. Write out the first four terms of each of the following geometric sequences, given the first term b_1 and common ratio q.
   (a) $b_1 = 1$ and $q = 3$
   (b) $b_1 = 1$ and $q = \frac{1}{2}$
   (c) $b_1 = -10$ and $q = \frac{1}{2}$
   (d) $b_1 = 27$ and $q = -\frac{1}{3}$

2. Calculate $S = 1 + 3 + 9 + 27 + 81 + 243$, first via the method of multiplying by the common ratio, then by plugging into the formula directly. Which method do you like better?

3. Calculate $S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$, using your preferred method.

4. What are the first two terms of the geometric sequence b_1, b_2, 24, 36, 54 ... ? Remember that you can find the common ratio by dividing a term by the previous term.

5. What is the geometric mean of 12 and 3?

6. What is the common ratio of the geometric sequence $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, ... ? What is b_{10}? b_{99}? b_{100}?

7. Calculate: $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + ... + \dfrac{1}{2^{10}}$

8. Calculate the sum $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + ... - 2^{15}$

***9.** Calculate the sum $1 - 2 + 2^2 + 2^3 - 2^4 + 2^5 + 2^6 - 2^7 + 2^8 + ... + 2^{14}$