MATH 10
ASSIGNMENT 20: CONTINUOUS FUNCTIONS

MAR 21, 2021

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called continuous if, for every sequence $a_n \in \mathbb{R}$ which has a limit: $\lim a_n = A \in \mathbb{R}$, the sequence $f(a_n)$ also has a limit and $\lim f(a_n) = f(A)$.

For example, function $f(x) = x$ is continuous, while the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not (see Problem 1 below)

Instead of functions $f\colon \mathbb{R} \to \mathbb{R}$, the same definition can be applied to functions between other sets: if X, Y are metric spaces (i.e., have sets with the notion of distance, satisfying all required properties), then the above definition works without any changes for functions $f\colon X \to Y$. For example, we can talk about continuous functions on an interval $[0, 1]$ or on the set $\mathbb{R}_+$ of positive real numbers. Note that in the latter case we only require that $f(a_n)$ converge for a sequence $a_n \in \mathbb{R}_+$ which has a limit also in $\mathbb{R}_+$. For example, we do not require that $f(1/n)$ converge.

Homework

1. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not continuous.

2. Prove that the function $f(x) = x^2 + 1$ is continuous.

3. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions.

(a) Prove that $f + g$, $f - g$, fg are also continuous. [Hint: remember the limit laws?]

(b) Prove that f/g is continuous on the set $X = \{x \in \mathbb{R} \mid g(x) \neq 0\} \subset \mathbb{R}$.

(c) Deduce that any polynomial is continuous everywhere on $\mathbb{R}$, and a rational function $f(x) = p(x)/q(x)$ is continuous everywhere it is defined.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that then the set $A = \{x \in \mathbb{R} \mid f(x) > 0\}$ is open in $\mathbb{R}$ (if you have forgotten the definition of open set, review Assignment 12 from January 6th). Hint: otherwise, set A contains a point $x \in \partial A$; then choose a sequence $x_n \in A'$ such that $\lim x_n = x$, where $A' = \{x \in \mathbb{R} \mid f(x) \leq 0\}$ is the complement of A.

***5.** Modify the previous proof to show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, and $U \subset \mathbb{R}$ is open, then $f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$ is also open.

***6.** Now show the converse: if, for any open set $U \subset \mathbb{R}$, $f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$ is also open, then $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.