SERIES

Recall: given a sequence a_n, we define

$$\sum_{i=1}^{\infty} a_i = \lim S_n, \qquad \text{where}$$

$$S_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

(if this limit exists; otherwise we say that the series diverges and expression $\sum_{i=1}^{\infty} a_i$ is meaningless). For example:

$$1 + r + r^2 + \cdots = \sum_{i=0}^{\infty} r^i = \lim \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}, \qquad |r| < 1$$

(this series is called the *geometric series*).

Note that it is quite possible that the sequence a_n converges but the sequence S_n of partial sums does not converge and thus the series $\sum_{i=1}^{\infty} a_i$ diverges!!

In the last HW, we have proved the following facts.

Theorem.

1. If a series $\sum a_n$ converges, then $\lim a_n = 0$. (*Converse is not true: even if $\lim a_n = 0$, the series may diverge*).

2. If $0 \le a_n \le b_n$, and $\sum b_n$ converges, then $\sum a_n$ also converges.

In fact, there is a more general result:

Theorem (Comparison test). *If a_n, b_n are sequences such that $b_n \ge 0$, $|a_n| \le b_n$ and the series $\sum_{1}^{\infty} b_n$ converges, then $\sum_{1}^{\infty} a_n$ also converges.*

The proof of this result will be given later.

HOMEWORK

1. A tortoise is moving on the plane starting at the origin and then going 1 unit along the positive direction of x axis; then turning $90°$ to the left and going for 0.9 units, then turning $90°$ to the left and going for $(0.9)^2$ units, then....

   (a) Show that if we consider the plane as the complex plane $\mathbb{C}$, then the position of the tortoise after n steps will be at the point $1 + r + r^2 + \cdots + r^{n-1}$, where $r = 0.9i$.

   (b) Find where the tortoise will end up in the limit, after infinitely many steps.

2. Let a_n be a sequence such that $r = \lim \frac{|a_{n+1}|}{|a_n|}$ exists.

   (a) Show that if $r > 1$, then $\lim a_n$ does not exists, and therefore $\sum_{1}^{\infty} a_n$ diverges (compare with Problem 1 from previous HW).

   (b) Prove that if $r < 1$, then the series $\sum_{1}^{\infty} a_n$ converges. [Hint: compare with geometric series.]

   (c) Give examples showing that if $r = 1$, then series $\sum_{1}^{\infty} a_n$ may converge or diverge.

   This is known as the *ratio test* for series convergence.

3. Prove that for any $x \in \mathbb{C}$, the series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges. [Hint: use the previous problem.]

4. Let $E(x)$ be as defined in the previous problem. Prove that then $E(x+y) = E(x)E(y)$. [Hint: both sides can be written as "double series" $\sum a_{m,n} x^n y^m$. You can use without a proof that in all the series involved, rearranging the terms in any order will not affect the value of the series.]

5. Let $e = \sum_{n=0}^{\infty} \frac{1}{n!} = E(1)$. Prove that then $E(x) = e^x$:
 (a) For all integer x
 (b) For all rational $x = p/q$
 *(c) For all real x