MATH 10
ASSIGNMENT 17: COMPLETENESS AXIOM
FEB 28, 2021

LEAST UPPER BOUND

Definition. A number M is called an *upper bound* of set S if for any $s \in S$, we have $s \leq M$.
A number M is called the *least upper bound* of set S (notation: $M = \sup(S)$) if

1. M is an upper bound of S, i.e. $\forall s \in S: \ s \leq M$
2. M is the smallest possible upper bound: if $M' < M$, then there exists $s \in S$ such that $s > M'$ and thus M' can not be an upper bound.

Note that it is possible that the least upper bound is not in S.
Condition (2) can be rewritten in this form:

$$\forall \varepsilon > 0: \ \exists s \in S: \ M - \varepsilon < s \leq M$$

which is sometimes more convenient.

Axiom (Completeness axiom). *For any set $S \subset \mathbb{R}$ which is bounded above, there exists the least upper bound.*

This is one of the defining properties of real numbers. There are many equivalent formulations of this property, such as Theorem 1 below or nested intervals property (see Problem 4). It is taken as an axiom of real numbers.
Note that this property fails for rational numbers: for example, set $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded above but has no least upper bound (in $\mathbb{Q}$). It does have a least upper bound in $\mathbb{R}$, namely $\sqrt{2}$.

LIMITS OF BOUNDED SEQUENCES

Theorem 1. *Any increasing bounded sequence has a limit.*

PROBLEMS

1. Compute the limits of the following sequences.
   (a) $\lim \dfrac{n^3 + 5n - 7}{(50n + 3)(2n - 7)}$
   (b) $\lim \dfrac{(-1)^n}{2^n}$

2. Find the least upper bound of the following sets (if they exist):
   (a) $S = [0, 1]$
   (b) $S = (0, 1)$
   (c) $\{1 - \frac{1}{n}\}$, $n = 1, 2, \dots$
   (d) $\{x \in \mathbb{R} \mid x^2 < 2\}$

3. Prove Theorem 1. [Hint: let $M = \sup\{a_n\}$. Show that then M is the limit.]

4. (a) Consider a sequence of nested intervals:

   $$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \dots$$

   Use completeness axiom to prove that then, there exists a point c which belongs to all of these intervals: for all n, $a_n \leq c \leq b_n$. Is such a point unique?
   [Hint: any of the b_i is an upper bound of set $S = \{a_1, \dots, a_k, \dots\}$.]
   (b) Show that the statement of the previous part fails if we replace closed intervals by open intervals (a_n, b_b). [Hint: consider intervals $(0, \frac{1}{n})$.]

5. Let

   $$a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n + 1)}$$

   (a) Compute a_1, a_2, a_3, a_4. Can you guess a general formula? [Hint: $\frac{1}{n \cdot (n+1)} = \frac{1}{n} - \frac{1}{n+1}$.]

(b) Find $\lim a_n$

(c) Let now
$$b_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

Use inequality $\frac{1}{(n+1)^2} \leq \frac{1}{n \cdot (n+1)}$ to prove that $b_n \leq a_{n-1} + 1$

(d) Prove that b_n has a limit. [This limit is actually equal to $\pi^2/6$, but it is rather hard to prove.]