

MATH 10
ASSIGNMENT 8: SYSTEMS OF LINEAR EQUATIONS CONTINUED
NOVEMBER 15, 2020

REVIEW OF LAST TIME

Recall that we can systematically represent a system of linear equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned}$$

in terms of its augmented matrix

$$A|\mathbf{b} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right].$$

Using elementary row transformations, such an augmented matrix can be brought to the ‘row echelon form’, where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\left[\begin{array}{cccccc|c} X & * & * & * & * & * & * \\ 0 & 0 & 0 & X & * & * & * \\ 0 & 0 & 0 & 0 & X & * & * \end{array} \right]$$

(here X ’s stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X ’s in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chosen arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$(1) \quad \begin{aligned}x_1 + x_2 + x_3 &= 5 \\x_2 + 3x_3 &= 6\end{aligned}$$

variables x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$\begin{aligned}x_2 &= 6 - 3x_3 = 6 - 3t \\x_1 &= 5 - x_2 - x_3 = -1 + 2t\end{aligned}$$

SYSTEMS WITH NO SOLUTIONS

It could happen that a system of linear equations has no solutions. For example, if the augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right]$$

then the second equation reads $0 \cdot x_1 + 0 \cdot x_2 = 1$, which clearly has no solutions. It happens if in the row echelon form, there is an row which has all zero entries except the last one (corresponding to the right hand side of the equation), which is non-zero.

LINES AND PLANES IN \mathbb{R}^3

As we saw before, a single linear equation $ax_1 + bx_2 + cx_3 = d$ describes a plane perpendicular to the vector (a, b, c) in \mathbb{R}^3 (we assume that at least one of a, b, c is non-zero). If we use our system of linear equation techniques to solve this equation, we obtain something of the form

$$\mathbf{x} = \mathbf{a} + t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2,$$

where $t_1, t_2 \in \mathbb{R}$ are parameters which can take any real values and $\mathbf{a}, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ are some fixed vectors found by solving the equation. This is the *parametric* definition of a plane.

Likewise, the parametric definition of a line in three-dimensional space is

$$\mathbf{x} = \mathbf{a} + t\mathbf{v},$$

for some vectors $\mathbf{a}, \mathbf{v} \in \mathbb{R}^3$.

HOMEWORK

- Find a polynomial $p(x)$ of degree 3 which satisfies the following conditions

$$p(0) = 5, \quad p(1) = 2, \quad p(-1) = 4, \quad p(3) = -40.$$

- Write the equation of a plane in \mathbb{R}^3 passing through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. [Hint: if the equation of the plane is $ax_1 + bx_2 + cx_3 = d$, then plugging in it each of the points gives a condition on a, b, c, d ; this gives a system of linear equations.]

Is such a plane unique?

- Let P be the plane in \mathbb{R}^3 described parametrically, as the set of all points of the form

$$\mathbf{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad t_1, t_2 \in \mathbb{R}.$$

Write an equation of this plane. [Hint: find 3 points on this plane.]

- Consider two planes $2x + 3y - z = 0$, $4x + z = 4$. Prove that their intersection is a line; write the line in the parametric form, by writing a generic point in the line as $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ for some vector \mathbf{v} .
- Find the intersection of 3 planes

$$x + 2y + 3z = 3$$

$$3x + y + 2z = 3$$

$$2x + 3y + z = 3$$