Class work 14. Algebra.

Algebra.

Math. Logic

A speech of a person or a text written on paper contain sentences. This is the way how we exchange the information between us. The information in every sentence can be a true fact, false, or sometime we just can't say is it true or false. For example, the sentence:
"The Earth is rotating around the Sun" is true.
"Paris is the capital of Germany" is false.
"Math is fun!" or "What time is it?" are the sentences we can't tell either it's true or false. Can you tell which sentence is "true", "false", or we can't tell:

a. "22 is an even number"
b. "44 is an odd number"
c. "1001 is a cool number!"

Let's define "a statement" as a sentence about which we can tell (sometime after difficult process of proving) either it is true or false. Such statement also can be called **a proposition.** For example, our first sentence "The Earth is rotating around the Sun" was proved to be true after hundreds of years of discussions. The second sentence, "Paris is the capital of Germany", can be proved wrong after we will check it in the dictionary (assuming that we never took geography class). As for the third example, how we can tell is 1001 a cool number? What is "cool"? for whom? Base on the definition, the sentence "22 is an even number" is a statement, and this is a true statement. "44 is an odd number" is also a statement, but the false one. "1001 is a cool number!" is not a statement at all.

1. Which of the following sentences are statements (proposition)?
   a. When is the first day of school this year?
   b. The 4^{th} of July is Independence Day.
   c. How beautiful is it!
   d. Washington, DC is a capital of the United States.
   e. The sum of five and three.
   f. Three times five is twenty-six.

2. Which of the following statements (propositions) are true, and which are false?
   a. There are 31 days in each January.
   b. There are 28 days in each February.
   c. Sunday is followed by Tuesday.
   d. There are 7 days in each week.
   e. There are 7 letters in the word "table"
   f. The sum of all single digit natural numbers is equal to 45.
   g. Every 3-digit natural number is grater then 100.

h. There is a greatest 5-digit natural number.
i. There is a greatest natural number.
j. There is a smallest natural number.

Propositional logic

Propositional logic is a mathematical system for reasoning about propositions and how they relate to one another. It helps us to see how the truth one proposition influences the truth of the other propositions.

Let's take a look at the statement "New York City is the capital of the United States". We, definitely, can say is it True or False. Of cause it's not true, we all know that the capital of the US is Washington, DC. So, we can say "it is not true, that New York City is the capital of the US", or, in a little more usual language, "New York City isn't the capital of the US". The last statement is a true statement.

"New York City is the capital of the United States"	False
"New York City isn't the capital of the US" (negation)	True

If the proposition is True, its negated version has to be False and vice versa. They can't be both True or both False. This rule of the propositional logic is one of the oldest and is called "The law of the excluded middle".

3. Let's try to construct negation of the several statements.

a. Number 111111111 is a prime number.
b. There is nothing on the table.
c. 0.5 and $\frac{1}{2}$ are not equale.
d. The area of a rectangle is equal to the product of its length and width.
e. Sum $18 \cdot 946 + 456$ is divisible by 9.
f. $45784 > 45784$
g. $345 < 12345$
h. All birds can flight.
i. All marine animals are fish.
j. Some students like math.
k. All natural numbers are divisible by 3.
l. Penguins live on the North Pole.
m. Polar bears live on the South Pole.

Each proposition can be represented by a variable, these variables can be small or capital letters. The value of such variables can be either **true** or **false**.
For example let's denote the proposition "The area of a rectangle is equal to the product of its length and width" is p, the negation will be denoted as $\neg p$.

Logical NOT: $\neg p$. Read "not p"

$\neg p$ is true if and only if p is false.

The relationship between propositions is called a logical connectives. For these connectives we can create a truth table.

p	$\neg p$
T	F
F	T

4. Using the law of the excluded middle prove, that the negation of statement was made incorrectly.

	Statement	**Negation**
1	All cats are gray.	All cats are not gray
2	Some berries are sweet.	Some berries are not sweet.
3	There are 30 days in some months.	There are no 30 days in some months.
4	All birds can fly.	There are no birds that can fly.

Categorical statements are the statements about the relationship between categories or classes of objects. It states whether one category is fully contained within another, is partially contained (there are at least one member of the category) within another, or is completely separate.

"Any (all) natural number is divisible by 3"

is a categorical statement and it is a false statement (category "natural numbers" is fully contained in another category "divisible by 3"). It is very easy to show: 5 is a natural number and it is not divisible by 3. It means that not any natural number is divisible by 3, some of the them are not disable by 3 and only one such example is enough to prove that the statement is false.

"The sum of any even numbers is an even number" is a categorical statement. The statement is about the category "sum of two arbitrary even numbers", this category belongs to the set of even numbers. We can ether prove it wrong by showing at least one example of the odd sum of two even number, or prove it true by reasoning. It is not enough to show several examples to prove that this statement is true and of course there are no example to prove it wrong.

Prove. Any even number can be represented as $2k$ (or $2n$) where $k, n \epsilon N$

$$2k + 2n = 2(k + n), \quad k, n \epsilon N$$

So, the sum is divisible by 2, or even number.

How the negation of the categorical statement can be done?
If the statement is about the whole category (all elements of the category) which forms a subset (we can use the set theory formalism here) of another category, the negated statement will show the existence of at least one element of the set, which doesn't belong to the category.

"All birds can fly. " – the statemen is telling us that the whole category (all birds) is belongs to another category, things that can fly. The negation of this statemen should tell us that there is at least one element (one kind of birds) which can't fly. We can formulate it as:
"Some birds can't fly" or as "There are birds that can't fly."

$q =$ "All men are not bald" , $\neg q$ = "Some men are bald" and vice versa:
$q =$ "Some cats are gray", $\neg q$ = "All cats are not gray"

Using logical connectives, we can create more complex propositions, for example logical **AND** connective:
***p*=I like math and physics. (I like math AND I like physics.)**

In which case this statement will be a true statement?

1. I like math, but I don't like physics.
2. I like physics, but I don't like math.
3. I like both, math and physics.
4. I don't like math, and I don't like physics.

To negate the statement, we have to make the negation which is false. What do you think such statement will look like?
In this case there are two statements, first is "I like math" (statement $\boldsymbol{a}$) and the second is "I like physics" (statement ***b***). Let's create a Truth table for our statement **A** AND **B**

a	***b***	$\boldsymbol{a \wedge b}$ **(AND, ·)**
T	T	T
T	F	F
F	T	F
F	F	F

The negation of each of the statements A and B are "I don't like math", "I don't like physics". Truth table of **¬A** (not A) OR **¬B** (not B). In this case the whole statement is true if at least one of the statements is true (or both statements are true).

$\neg a$	$\neg b$	$\neg a \vee \neg b$ **(OR, +)**
T	T	T
T	F	T
F	T	T
F	F	F

From this we can see that the negation of the statement “I like math and physics” should be true in the case when the negation of one of the two statements or both are true. We came up with another logical symbol (logical OR).

$$\boldsymbol{p = a\wedge \mathbf{b}}; \qquad \boldsymbol{\neg p = a\vee b}$$

Logical OR

Statemen *q* = ”I eat pear or apple after diner”
Statemen *a* = “I eat pear after diner”, statemen *b* = ”I eat apple after diner”.

a	***b***	***a* OR *b* (∨)**
T	T	T
T	F	T
F	T	T
F	F	F

In real life you will probably say “I eat ether pear or apple after diner”. The phrase “either…or” in language usually means one out of two possibilities, not both, but in math logic the statement “A or B” is true if A is true, B is true, both A and B are true”.

Exercises:

1. Which of the following statements are categorical statements?
    a. Some type of plants and animals are listed in the list of endangered species.
    b. All planets of the Solar system are rotating around the Earth in the same direction.
    c. Some butterflies are yellow.
    d. There are 22 books on the shelf.
    e. Any natural number is greater than 0.

2. Find one counter example to prove the following statements wrong:
    a. All natural numbers are greater then 1.
    b. Any number divisible by 5 ends with digit 5.
    c. All rivers of the united states flow into the Pacific Ocean.
    d. All marine animals are fish.
    e. All American cities lie south of 50° latitude line.

3. Mother told Mary, that she can play videogames if she will do her homework, also will do her room, and will do dishes after diner. Will Mary play videogames if she
    a. Did her room?
    b. Did her room and dishes?
    c. Did her homework?
    d. Did her homework and dishes?
    e. Did all three assignments?

4. On the other day mother told Mary that she will play videogames if she will do her homework or will do dishes. Will Mary play the videogames if she
   a. Did her homework?
   b. Did the dishes?
   c. Did both assignment?

5. The following statements are proven. Can you tell which prove is right and which is wrong?
   a. All natural numbers are divisible by 7, for example: 14:7=2.
   b. Some proper fractions have denominator equals to 8, for example: denominator of the fraction $\frac{3}{8}$ is 8.
   c. There are even numbers multiple of 3, for example 36 is multiple of 3, 36:3=12
   d. Some nouns in English language contain 5 letters, for example "table".
   e. All verbs in English language start with letter "w", for example " (to) write".
   f. There are books written by Joanne Rowling, for example *Harry Potter and the Philosopher's Stone* .

6. Inhabitants of the city A always say the true, inhabitants of the city B always lie, and inhabitant of the city C say true or lie every other time. Fire department got a call from somebody: "We got a fire here, come over as soon as you can". "Where is the fire?", - the fireman asked. "In the city B", - was the answer. Where should they go, if the fire in one of the cities is real?

   Many of the questions of this assignment refer to the famous (among logic puzzle fans) island of Knights and Knaves. On this island, there are two kinds of people: Knights, who always tell the truth, and Knaves, who always lie. Unfortunately, there is no easy way of knowing whether a person you meet is a knight or a knave. . .
   Copyright notice: a lot of these problems come from books of Raymond Smullyan. If you liked them, get his books in the library and you will find there many more puzzles of the same sort. I would especially recommend The Lady or the Tiger? and The Riddle of Scheherazade. You can also find a number of such puzzles online at http://philosophy.hku.hk/think/logic/puzzles.php.

7. You meet two inhabitants: Peggy and Zippy. Peggy tells you that 'of Zippy and I, exactly one is a knight'. Zippy tells you that only a knave would say that Peggy is a knave.
   Can you determine who is a knight and who is a knave?

8. You meet two inhabitants: Marge and Zoey. Marge says, 'Zoey and I are both knights or both knaves.' Zoey claims, 'Marge and I are the same.'
   Can you determine who is a knight and who is a knave?

9. You meet two inhabitants: Ted and Zeke. Ted claims, 'Zeke could say that I am a knave.' Zeke claims that it's not the case that Ted is a knave.
   Can you determine who is a knight and who is a knave?

10. You meet two inhabitants: Ned and Zoey. Zed says that it's false that Zoey is a knave. Zoey claims, 'I and Ned are different.'

Can you determine who is a knight and who is a knave?

11. You meet two inhabitants: Sue and Marge. Sue says that Marge is a knave. Marge claims, ‘Sue and I are not the same.’
Can you determine who is a knight and who is a knave?

12. On the island of knights and knaves, you meet two inhabitants: Zoey and Mel. Zoey tells you that Mel is a knave. Mel says, “Neither Zoey nor I are knaves.” So who is a knight and who is a knave?

13. You meet two inhabitants: Betty and Peggy. Betty tells you that Peggy is a knave. Peggy tells you, ‘Betty and I are both knights.’
Can you determine who is a knight and who is a knave?

14. You meet two inhabitants: Zed and Peggy. Zed says that Peggy is a knave. Peggy tells you, ‘Either Zed is a knight or I am a knight.’
Can you determine who is a knight and who is a knave?

15. You meet two inhabitants: Zed and Alice. Zed tells you, ‘Alice could say that I am a knight.’ Alice claims, ‘It’s not the case that Zed is a knave.’
Can you determine who is a knight and who is a knave?

16. While visiting the Knights and Knaves Island, you pass a beautiful garden where three islanders, Sam, Bob, and Tom are watching the sunset. You ask Sam, ”Are you a knight or a knave?” Sam is shy; you cannot hear his quiet answer. So you ask Bob, ”What did Sam say?” Bob answers, ”He said that he is a knave.” ”Don’t trust Bob! Bob is a knave!” screams Tom.
Can you decide whether Bob and Tom are knights or knaves?

13. A traveler to the island of Knights and Knaves meets a group of five people (call them A, B, C, D, E).
A says: “exactly one of us is a Knight”
B says: “exactly two of us are Knights”
C says: “exactly three of us are Knights”
D says: “exactly four of us are Knights”
E says: “all five of us are Knights”
Can you find out which of them are Knights?
14. You are in a maze on the island of knights and knaves. There are two doors: you know that
one leads to freedom and one leads to certain doom. There are two guards nearby, and you happen to know that one is a knight and one is a knave, but you don’t know who is who. They allow you to ask one of them a single question before you choose a door — what do you ask?