Class work 5. Algebra.

Algebra.

1. $2 + a$ and $35 - b$ are both divisible by 11. Prove that $a + b$ is divisible by 11 as well.

2. Simplify the following fractions:

$\frac{135+315}{15}$, $\frac{49+84}{77}$, $\frac{am+5a}{m+5}$,

$\frac{252+168}{132-12}$, $\frac{5kx-5xa}{k-a}$, $\frac{35b}{7ba+7bx}$,

Properties of exponent.

Let us explore the definition of the exponent, a^m, when m is a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times} \quad (1)$$

Based on this definition (1) we can show that

$$a^n \cdot a^m = \underbrace{a \cdot a \dots \cdot a}_{n\ times} \cdot \underbrace{a \cdot a \dots \cdot a}_{m\ times} = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n+m\ times} = a^{n+m} \quad (2)$$

and

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot \dots \cdot a^n}_{m\ times} = \underbrace{\underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times} \cdot \dots \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times}}_{m\ times} = a^{n \cdot m} \quad (3)$$

If we want to multiply $a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$ by another a we will get the following expression:

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \dots \cdot a}_{n+1\ times} = a^{n+1} = a^n \cdot a^1 \quad (4)$$

In order to have the set of power properties consistent, $a^1 = a$ for any number a.

We can multiply any number by 1, this operation will not change the number, so if

$$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0 \quad (5)$$

In order to have the set of properties of exponent consistent, $a^0 = 1$ for any number a.

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \dots \cdot a}_{m\ times}}, n > m$$

$$\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n-m\ times} = a^{n-m} \quad (6)$$

We can rewrite the expression (6) as

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \Big(\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}\Big) : \Big(\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}\Big) = a^n : a^m = a^{n-m} \quad (7)$$

We see that based on the definition of the exponentiation *n*-th and *m*-th powers add up if a^n and a^m are multiplied and subtract if a^n is divided by a^m. We know, on the other hand, that division is a multiplication by inverse number.

Let rewrite (6) again

$$\frac{a^n}{a^m} = a^n : a^m = a^{n+(-m)} = a^n \cdot \frac{1}{a^m} = a^n \cdot a^{-m} \ , a \neq 0 \ \text{for any } n \text{ and } m \in \mathbb{N}$$

Negative power of a number, not equal to 0, can be defined as

$$a^{-m} = \frac{1}{a^m} \ , \ a \neq 0$$

So, all properties of the exponentiation can be defined as:

1. $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a$, for any a
5. $a^0 = 1$, for any a
6. $\frac{1}{a^m} = a^{-m}$, for any $a \neq 0$
7. $(a \cdot b)^n = a^n \cdot b^n$

Also, if there are two numbers a and b:

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \ldots \cdot (a \cdot b)}_{n\ times} = \underbrace{a \cdot \ldots \cdot a}_{n\ times} \cdot \underbrace{b \cdot \ldots \cdot b}_{n\ times}$$

$$= a^n \cdot b^n \quad (9)$$

Last property is

$$(a \cdot b)^n = a^n \cdot b^n$$

I. A positive number raised into any power will result a positive number.

- Because the product of any number of positive factors gives as an outcome a positive number, and exponentiation is a product of same factors, we proved the statement.

II. A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

- Even number m of negative factors produces $\frac{m}{2}$ pairs of multiplied negative numbers each of which is positive, as we know, of product any quantity of positive factors is positive.
- For an odd number n, $\frac{n-1}{2}$ pairs of negative factors will be multiplying by one more negative number, so the whole product will be negative.

Exercises:

1. Compute:
   a. $(-3)^3$;
   b. -3^3;
   c. 2^7;
   d. $(-2)^7$;
   e. -2^7;
   f. $(2 \cdot 3)^3$;
   g. $2 \cdot 3^3$;
   h. $\left(\frac{1}{3}\right)^2$;
   i. $\frac{1}{3^2}$;
   j. 3^{-2};
   k. $(-3)^{-2}$;
   l. $(-5 \cdot 2)^3$

   Remember, that $a^n : a^m = a^{n-m} = a^{n+(-m)} = a^n \cdot \frac{1}{a^m} = a^n \cdot a^{-m}$

2. Write the following expressions in a shorter way replacing product with power:

   Examples:

   $(-a) \cdot (-a) \cdot (-a) \cdot (-a) = (-a)^4$, $\quad 3m \cdot m \cdot m \cdot 2k \cdot k \cdot k \cdot k = 6m^3k^4$

   $(-y) \cdot (-y) \cdot (-y) \cdot (-y)$;

   $(-5m)(-5m) \cdot 2n \cdot 2n \cdot 2n$;

   $-y \cdot y \cdot y \cdot y$;

   $-5m \cdot m \cdot 2n \cdot n \cdot n$;

   $(ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab)$;

   $p - q \cdot q \cdot q \cdot q \cdot q$;

   $a \cdot b \cdot b \cdot b \cdot b \cdot b$;

   $(p-q) \cdot (p-q) \cdot (p-q) \cdot (p-q)$;

3. Write the following expressions replacing exponent with a product of several factors:

   Examples: $(-x)^3 = (-x) \cdot (-x) \cdot (-x)$; $\quad 3y - a^4 = 3y - a \cdot a \cdot a \cdot a$

   $(-n)^3$; $\quad (-mn)^4 \quad 2x - y^3$

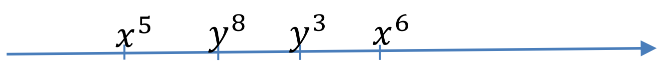
   $-x^2$; $\quad -mn^4 \quad (a+3b)^2$

   $(2c)^2$; $\quad 2c^2$; $\quad (2x-y)^3$

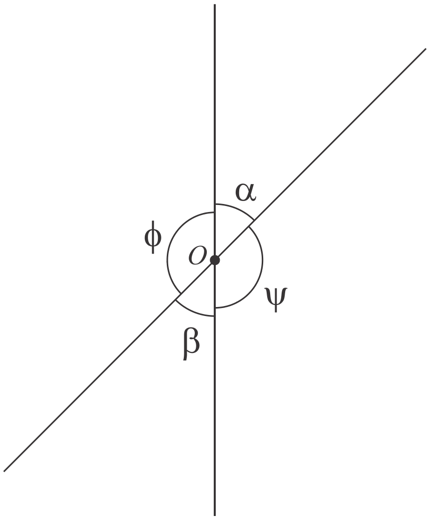
4.

$$x^5 < y^8 < y^3 < x^6$$

Where 0 should be placed?



Geometry.



α and β and ϕ and ψ are 2 pairs of vertical angles.

Vertical angles theorem:

Vertical angles are equal.

In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements. According to a historical legend, when Thales visited Egypt, he observed that whenever the Egyptians drew two intersecting lines, they would measure the vertical angles to make sure that they were equal. Thales concluded that one could prove that vertical angles are always equal and there is no need to measure them every time.

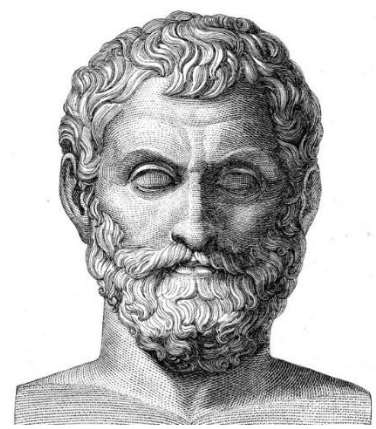
Proof:

$\angle\phi + \angle\alpha = 180°$ because they are supplementary by construction.

$\angle\phi + \angle\beta = 180°$ because they are supplementary also by construction.

$\Rightarrow \angle\alpha = \angle\beta$, therefore, we proved that if 2 angles are vertical angles then they are equal. Can we tell that invers is also the truth? Can we tell that if 2 angles are equal than they are vertical angels?

(**Thales of Miletus** 624-546 BC was a Greek philosopher and mathematician from Miletus. Thales attempted to explain natural phenomena without reference to mythology. Thales used geometry to calculate the heights of pyramids and the distance of ships from the shore. He is the first known individual to use deductive reasoning applied to geometry, he also has been credited with the discovery of five theorems. He is the first known individual to whom a mathematical discovery has been attributed (Thales theorem).



Exercises.

3. 4 angles are formed at the intersection of 2 lines. One of them is 30°. What is the measure of 3 others?