Class work 3. Algebra.

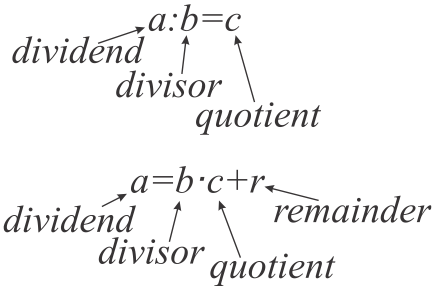
Algebra.

1. Divisibility.

We say that a natural number is divisible by another natural number if the result of this operation is a natural number. If this is not the case then we can divide a number with a remainder.
If a and n are natural numbers, the result of a division operation of $a \div n$ will be a quotient c, such that

$$a = b \times c + r$$

Where r is a remainder of a division $a \div b$. If r is 0, then we can tell that a is divisible by b.



- If we want to divide m by 15, what numbers we can get as a remainder?

If the remainder is 0, then quotient and divisor are both factors of dividend, $a = b \cdot c$, and to divide a number $\boldsymbol{a}$ by another number, $\boldsymbol{b}$, means to find such number $\boldsymbol{c}$, that $c \cdot b$ will give us a. So, because the product of 0 and any number is 0, than there is no such arithmetic operation as division by 0.

2. Divisibility rules.

A statement (or proposition) is a sentence that is either true or false, but not both. So '3 is an odd integer' is a statement.
But 'π is a cool number' is not a (mathematical) statement.
Note that '4 is an odd integer' is also a statement, but it is a false statement.
Are these sentences statements or not? If yes, are they true or false? Can you prove it?

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- The sum of 2 odd natural number is an even number
- Would you like some cake?
- $3 + x = 12$
- The sum of two even numbers.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $7 + 3 = 10$
- All birds can fly.

The rule of divisibility by 2 is:
If and only if the last digit of a number is an even number or 0 (0, 2, 4, 6, or 8) the number is even number (divisible by 2).

Proof of the divisibility by 2 rule:

Any natural number can be written as a sum:

$$\ldots + 1000 \cdot n + 100 \cdot m + 10 \cdot l + k = \cdots + 2 \cdot 500 \times n + 2 \cdot 50 \times m + 2 \cdot 5 \cdot l + k$$

Were n, m, l, and k are numbers of thousands, hundreds, tens, and units. If *k* is an even number or 0, it also can be represented as a product of 2 and another single digit number. Then the number can be written as:

$\ldots + 1000 \times n + 100m + 10 \times l + k = \cdots + 2 \times 500 \times n + 2 \times 50 \times m + 2 \times 5 \times l + 2 \times p$

(*p* can be 0, 1, 2, 3, and 4. Do you know why?). Distributive property is allowing us to represent this expression as a product:

$$\ldots + 1000 \times n + 100m + 10 \times l + k = \cdots + 2 \times 500 \times n + 2 \times 50 \times m + 2 \times 5 \times l + k$$
$$= 2 \times (\ldots + 500 \times n + 50 \times m + 5 \times l + p)$$

Now we can see that the number is divisible by 2 if its last digit is even or 0.

All other divisibility rules can be proved as well.

Factorization.

In mathematics factorization is a decomposition of one number into a product of two or more numbers, or representation of an expression as a product of 2 or more expressions, which called 'factors'. For example, we can represent the expression $a \cdot b + a \cdot c$ as a product of a and expression $(b + c)$. Can you explain why?

$$a \cdot b + a \cdot c = a \cdot (b + c)$$

Or in a numerical expression:

$$7 \cdot 5 + 7 \cdot 3 = 7 \cdot (5 + 3)$$

Or a number can be representing as product of two or more other numbers, for example:

$$40 = 4 \cdot 10 = 4 \cdot 2 \cdot 5, \quad 36 = 6 \cdot 6 = 3 \cdot 2 \cdot 6$$

Does any natural number can be represented as a product of 2 or more numbers besides 1 and itself? Natural numbers greater than 1 that has no positive divisors other than 1 and itself are called prime numbers.

Even numbers are the numbers divisible by 2 (they have 2 as a divisor), so they can be factorized as 2 times something else. Can an even number be a prime number? Is there any even prime number?

Prime factorization or integer **factorization** of a number is the determination of the set of **prime** numbers which multiply together to give the original integer. It is also known as **prime** decomposition.

168	2	180	2
84	2	90	2
42	2	45	3
21	3	15	3
7	7	5	5
1		1	

Prime factorization process:

Prime factors of 168 are $2, 2, 2, 3, 7$ and prime factors of 180 are $2, 2, 3, 3, 5,$

$$2 \times 2 \times 2 \times 3 \times 7 = 168; \quad 2 \times 2 \times 3 \times 3 \times 5 = 180$$

Eratosthenes proposed a simple algorithm for finding prime numbers. This algorithm is known in

mathematics as the Sieve of Eratosthenes.

In mathematics, the sieve of Eratosthenes, one of a number of prime number sieves, is a simple, ancient algorithm for finding all prime numbers up to any given limit. It does so by iteratively marking as composite, *i.e.*, not prime, the multiples of each prime, starting with the multiples of 2.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Exercises:

1. Proof that the sum of two any even natural numbers is an even number.
2. The remainder of 1932 ÷ 17 is 11, the remainder of 261 ÷ 17 is 6. Is 2193 = 1932 + 261 divisible by 17? Can you tell without calculating? Explain.
3. Find all natural numbers such that upon division by 7 the quotient and remainder will be equal.
4. Even or odd number will be the sum and the product of
   a. 2 odd numbers
   b. 2 even numbers
   c. 1 even and 1 odd number
   d. 1 odd and 1 even number

   Can you explain why? (a few examples do not prove the statement).

5. $a + 1$ is divisible by 3. Prove that $4 + 7a$ is divisible by 3 as well.

6. $2 + a$ and $35 - b$ are both divisible by 11. Prove that $a + b$ is divisible by 11 as well.

7. Find all prime numbers p such that p and $5p + 1$ both are prime numbers.

8. Even or odd number will be the sum
   $1 + 2 + 3 + \ldots + 10$
   $1 + 2 + 3 + \ldots + 100$
   $1 + 2 + 3 + \ldots + 100$

9. Can you say which of the following statements are true and which are false?
   a. If the natural number is divisible by 3 and 5, it's divisible by 15
      (if $a \vdots 3$ and $a \vdots 5 \Rightarrow a \vdots 15$)
   b. If the natural number is divisible by 15, it's divisible by 3 and 5.
      (if $a \vdots 15 \Rightarrow a \vdots 3$ and $a \vdots 5$)
   c. If natural number b is even, then $3b$ is divisible by 6. (if $b \vdots 2$, then $3b \vdots 6$)

10. Can the expression below be a true statement, if letters are replaced with numbers from 1 to 9 (different letters correspond to different numbers).

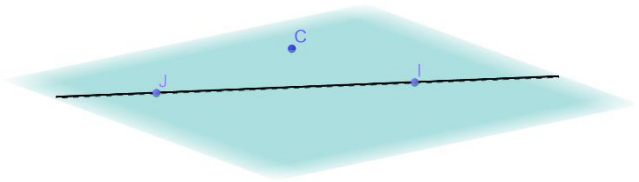
$$f \cdot l \cdot y = i \cdot n \cdot s \cdot e \cdot c \cdot t$$

Geometry.

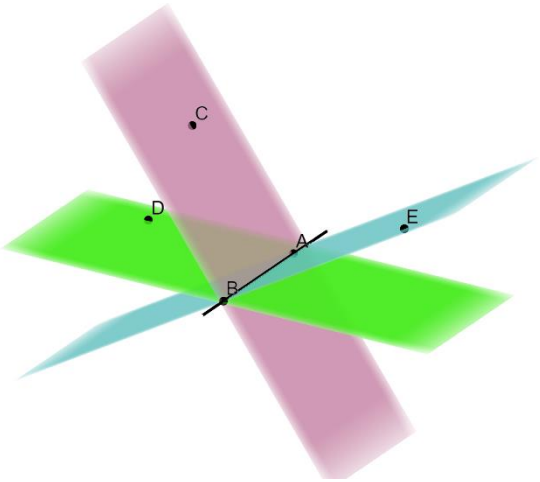
Points, lines, and plane.

There are two possibilities of mutual location of the line and the point on the plane: a point lies on a line or a point doesn't lie on the straight line. If 2 lines have 2 common points these lines coincide (only one straight line can pass through two points, it means, that two points define a line). Three points not lying on the same line, define a plane: only one plane can pass through three such points.

- How many planes can go through three point lying on the same line?



So, any line and a point outside of it, lie in the same plane. (We always can mark two points on the line and the third one will define this plane.)



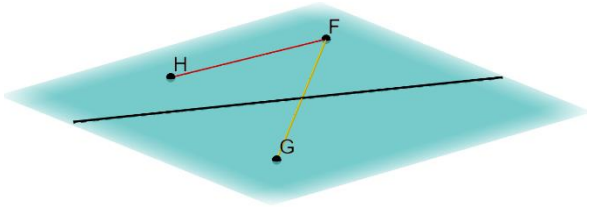
Two straight line can intersect (then they have one common point) or they can be parallel. In both cases they are on the same plane.

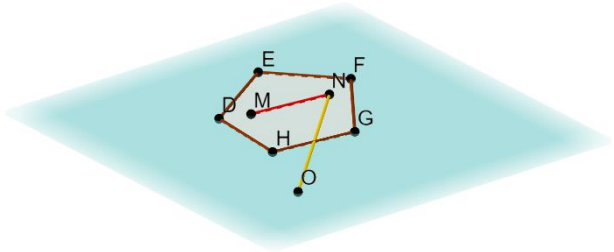
On a plane mark four points. How many lines you can draw through these four points. Look for all possible solutions.

Parallel lines are lines in a plane which do not meet; that is, two lines in a plane that do not intersect or touch each other at any point are said to be parallel.

- How two lines can be not parallel and do not intersect?
- The intersection of two lines is a point. What is the intersection of a plane and a line which do not belong to the same plane?
- What is an intersection of two planes?

Each straight line divides a plane into two domains. In these domains any two points on one side of the line may be connected without crossing the line itself and any two points on the two different sides of the lane can't be connected without crossing the line.





Enclosed area on a plane is the area limited by a closed curved line (or chain of line segments) any 2 points of which can be connected without crossing the curved line (or series of line segments) and any point inside of the limit can't be connected with any point outside of the limit without crossing the curved line (or chain of line segments).

Exercises.

1. How many planes are passing through two given point?
2. There are a plane p and a square ABCD. Is it possible that only one vertex of the square belongs to the plane? Only two vertices? Only three vertices belong to the plane? All four vertices?
3. How many lines can be drown through pairs of points A, B, C, and D if they ar on the same plane? If they are not on the same plane?