May 17, 2020

The final Math battle 9.

1. You have 100 beads, all different. How many different necklaces of length 50 can you make using these beads? The clasp on the necklace can be ignored, in the sense that 1-2-clasp-3-4 is the same as 1-clasp-2-3-4.
2. For a rectangle with the diagonal of fixed length, d, find, using Math 9 and below (no calculus),
   a. What is the largest possible area?
   b. What is the largest possible perimeter?
3. Is the product, $(2021 \cdot 2022 \cdot 2023 \cdot \ldots \cdot 4040)$, divisible by 2020! (2020 factorial)? What about $(2020 \cdot 2021 \cdot 2022 \cdot \ldots \cdot 4039)$?
4. Does there exist a polynomial $P(x)$ with integer coefficients such that $P(6) = 5$ and $P(14) = 9$?
5. Let $F(x)$ be a function obtained by a sequential application of function $f(x)$ 2020 times, $F(x) = \overbrace{f(f(f(\ldots f(f(x))\ldots)))}^{2020\ times}$. Find all real solutions of the equation:
   a. $F(x) = 2$, where the function $f(x) = \sqrt{\frac{1}{2}x^2 + 2}$
   b. $F(x) = 0$, where the function $f(x) = x^2 + 10x + 20$
6. Given an equilateral triangle ABC find all points M on the plane such that both triangles ABM and ACM are isosceles.

Extra problems.

1. How many natural numbers <1000 are not divisible by 7, 9 and 13?
2. Four friends, A., B., C. and D., decided to exchange presents. They agreed that each one prepares a present, which will then be randomly drawn. Hence, each can get, with equal probability, any of the four presents. What is the probability that no one gets his/her own present, while A. gets the present from D.?
3. A city has 10 bus routes. Is it possible to arrange the routes and the bus stops so that if one route is closed, it is still possible to get from any one stop to any other (possibly changing the route along the way), but if any two routes are closed, there are at least two stops such that it is impossible to get from one to the other?
4. In the number 454**, find the missing digits so that the number is divisible by 2, by 7, and by 9.

5. Prove that the product of any m consecutive integer numbers is divisible by $m!$
6. Find all real solutions of the equation $\overbrace{f(f(f(\ldots\ldots(f(x))\ldots\ldots)}^{2020\ times}= 0$ where the function $f(x) = x^2 + 10x + 20$ is sequentially applied 2020 times.