

Homework for April 26, 2020.

### Algebra. Complex numbers.

Please, complete the previous homework assignments from this year. Review the classwork handout on complex numbers. Complete the classwork exercises and solve the following problems.

#### Problems.

1. Compute:

- $(2 - i)^{-1}$
- $\frac{-i}{4\sqrt{3}-i}$
- $\frac{1}{3-4i}$
- $(1 + i)^{-10}$

2. Solve the following equations in complex numbers:

- $z^2 = -i$
- $z^2 = 2\sqrt{3} + 2i$
- $z^2 - z - 1 = 0$
- $z^2 + z - 1 = 0$

### Geometry. Vectors.

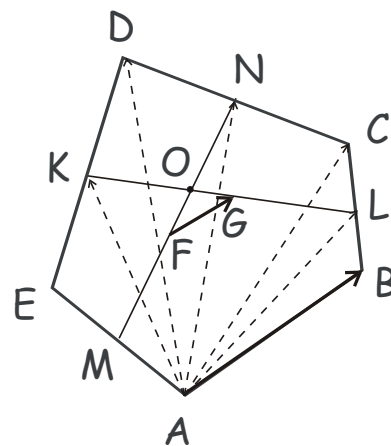
Please, complete problems from the previous homework assignment. Review the classwork handout on vectors. Solve the following problems.

#### Problems.

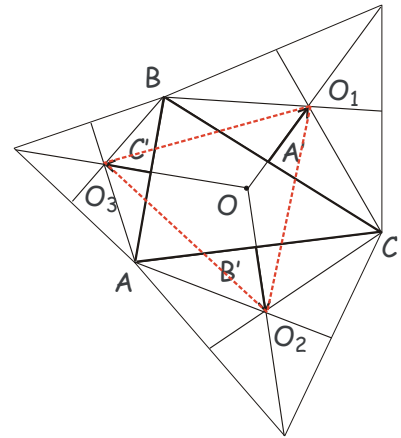
- In a pentagon  $ABCDE$ ,  $M$ ,  $K$ ,  $N$  and  $L$  are the midpoints of the sides  $AE$ ,  $ED$ ,  $DC$ , and  $CB$ , respectively.  $F$  and  $G$  are the midpoints of thus obtained segments  $MN$  and  $KL$  (see Figure). Show that the segment  $FG$  is parallel to  $AB$  and its length is  $\frac{1}{4}$  of that of  $AB$ ,  $|FG| = \frac{1}{4}|AB|$ .

Hint: use the results of one of the previous problems, expressing the median of a triangle via adjacent sides.

- Three equilateral triangles are erected externally



on the sides of an arbitrary triangle  $ABC$ . Show that triangle  $O_1O_2O_3$  obtained by connecting the centers of these equilateral triangles is also an equilateral triangle (Napoleon's triangle, see Figure).



3. If you have not done it yet, solve the following problem from the last homework. Vectors  $\overrightarrow{AA'}$ ,  $\overrightarrow{BB'}$  and  $\overrightarrow{CC'}$  are represented by the internal bisectors in the triangle  $ABC$ , directed from each vertex to the point on the opposite side.

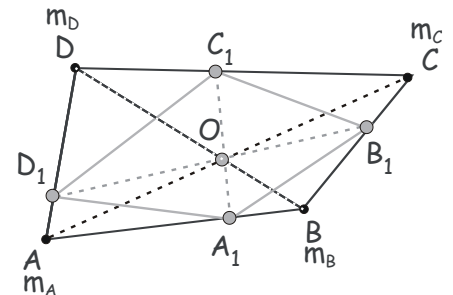
Express the sum,  $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$  through vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  (and the sides of the triangle,  $|AB| = c$ ,  $|BC| = a$ ,  $|CA| = b$ ). For what triangles  $ABC$  does this sum equal 0?

4. Let  $A$ ,  $B$  and  $C$  be angles of a triangle  $ABC$ .

a. Prove that  $\cos A + \cos B + \cos C \leq \frac{3}{2}$ .

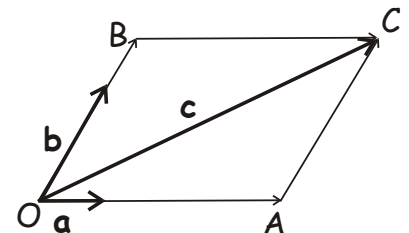
b. \*Prove that for any three numbers,  $m, n, p$ ,  $2mncos A + 2np cos B + 2pm cos C \leq m^2 + n^2 + p^2$

5. \*A quadrilateral  $A_1B_1C_1D_1$  is inscribed in the quadrilateral  $ABCD$  in such a way that diagonals of both quadrilaterals intersect at the same crossing point,  $O$  (see Figure). Show that this is possible if  $\frac{|AA_1|}{|A_1B|} \frac{|BB_1|}{|B_1C|} \frac{|CC_1|}{|C_1D|} \frac{|DD_1|}{|D_1A|} = 1$ .

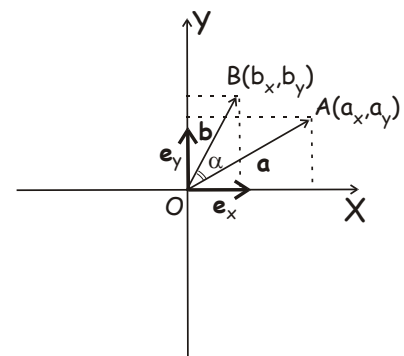


6. Prove that if vectors  $\vec{a}$  and  $\vec{b}$  satisfy  $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$ , then  $\vec{a} \perp \vec{b}$ .

7. Show that for any two non-collinear vectors  $\vec{a}$  and  $\vec{b}$  in the plane and any third vector  $\vec{c}$  in the plane, there exist one and only one pair of real numbers  $(x, y)$  such that  $\vec{c}$  can be represented as  $\vec{c} = x\vec{a} + y\vec{b}$ .



8. Derive the formula for the scalar (dot) product of the two vectors,  $\vec{a}(x_a, y_a)$  and  $\vec{b}(x_b, y_b)$ ,  $(\vec{a} \cdot \vec{b}) = x_ax_b + y_ay_b$ , using their representation via two perpendicular vectors of



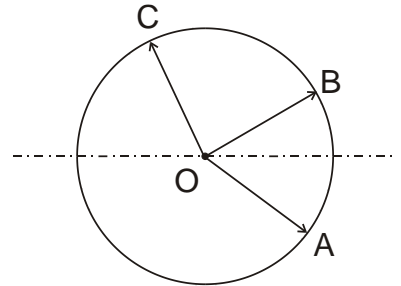
unit length,  $\vec{e}_x$  and  $\vec{e}_y$ , directed along the  $X$  and the  $Y$  axis, respectively.

9. Given vectors  $\vec{a}$  and  $\vec{b}$ , show that vector  $\vec{a} - \frac{1}{b^2}(\vec{a} \cdot \vec{b})\vec{b}$  is perpendicular to  $\vec{b}$ .

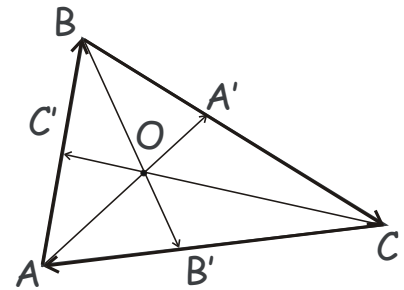
10. Vectors  $\vec{OA}$ ,  $\vec{OB}$  and  $\vec{OC}$  are represented by the radial segments directed from the centre  $O$  of the circle to points  $A$ ,  $B$  and  $C$  on the circle (see Figure). What are the angles  $\angle AOB$ ,  $\angle AOC$  and  $\angle COB$ , if

a.  $\vec{OC} = \vec{OA} - \vec{OB}$

b.  $\vec{OC} = \vec{OA} + \vec{OB}$



11. Vectors  $\vec{AA'}$ ,  $\vec{BB'}$  and  $\vec{CC'}$  are represented by the internal bisectors in the triangle  $ABC$ , directed from each vertex to the point on the opposite side (see figure). Express the sum,  $\vec{AA'} + \vec{BB'} + \vec{CC'}$  through vectors  $\vec{AB}$  and  $\vec{AC}$  (and the sides of the triangle,  $|AB| = c$ ,  $|BC| = a$ ,  $|CA| = b$ ). For what triangles  $ABC$  does this sum equal 0?



12. Given triangle  $ABC$ , find the locus of points  $M$  such that  $(\vec{AB} \cdot \vec{CM}) + (\vec{BC} \cdot \vec{AM}) + (\vec{CA} \cdot \vec{BM}) = 0$ . Using this finding, prove that three altitudes of the triangle  $ABC$  are concurrent (i.e. all three intersect at a common crossing point, the orthocenter of the triangle  $ABC$ ).
13. Let  $O$  be the circumcenter (a center of the circle circumscribed around) and  $H$  be the orthocenter (the intersection point of the three altitudes) of a triangle  $ABC$ . Prove, that  $\vec{HA} + \vec{HB} + \vec{HC} = 2\vec{HO}$ .