Homework for April 5, 2020.

Algebra.

Review the classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Solve the following problems.

1. Let x_1, x_2 and x_3 be distinct real numbers. Prove that there exists a unique polynomial, $P(x)$, of degree 2 such that $P(x_1) = 1$, $P(x_2) = P(x_3) = 0$. [Hint: if $P(x_1) = 0$, then $P(x)$ is divisible by $(x - x_1)$.] Find this polynomial if $x_1 = 2, x_2 = -1, x_3 = 5$.
2. As before, let x_1, x_2 and x_3 be distinct real numbers, and let y_1, y_2 and y_3 be any collection of numbers. Prove that there is a unique quadratic polynomial $f(x)$ such that $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3$. Find this polynomial if $x_1 = 2, x_2 = -1, x_3 = 5, y_1 = 3, y_2 = 6, y_3 = 18$. [Hint: look for in the form $f(x) = y_1 f(x_1) + \cdots$.]
3. Prove the following general result: given numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$, such that x_i are distinct, there exists a unique polynomial $f(x)$ of degree $n - 1$ such that $f(x_i) = y_i, i = 1, \ldots, n$. (For $n = 2$, this is a statement that there is a unique line through two given points.)
4. Prove that if $P(x)$ is a polynomial with integer coefficients, then for any integer a, b, the difference $P(a) - P(b)$ is divisible by $a - b$.
5. Le t x_1 and x_2 be the roots of the polynomial, $x^2 + 7x - 3$. Find
   a. ${x_1}^2 + {x_2}^2$
   b. $\frac{1}{x_1} + \frac{1}{x_2}$
   c. $(x_1 - x_2)^2$
   d. ${x_1}^3 + {x_2}^3$

Algebra/Trigonometry.

Please, complete the previous homework assignments from this year. Review the classwork handout on trigonometric functions. Additional reading on trigonometric functions is Read Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163), http://en.wikipedia.org/wiki/Trigonometric_functions http://en.wikipedia.org/wiki/Sine. Solve the following problems.

Problems.

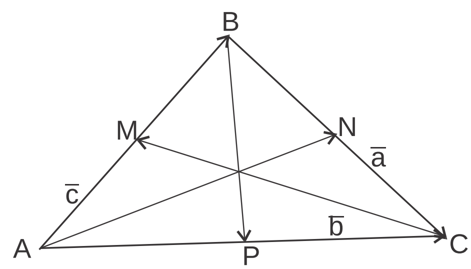
1. Simplify the following expressions:
   a. $\frac{\sin(\pi+\alpha)\cos(\pi-\alpha)}{\sin(\alpha-\pi)\cos(\alpha+\pi)}$
   b. $\frac{\cot^2\left(\alpha+\frac{\pi}{2}\right)\cos^2\left(\alpha-\frac{\pi}{2}\right)}{\cot^2\left(\alpha-\frac{\pi}{2}\right)-\cos^2\left(\alpha+\frac{\pi}{2}\right)}$
   c. $\frac{\cot\left(\frac{3\pi}{2}-\alpha\right)}{1-\tan^2(\alpha-\pi)}\cdot\frac{\cot^2(2\pi-\alpha)-1}{\cot(\alpha+\pi)}$
   d. $\frac{\cos^2\left(\alpha-\frac{3\pi}{2}\right)}{\sin^{-2}\left(\alpha+\frac{\pi}{2}\right)-1}\cdot\frac{\sin^2\left(\alpha+\frac{3\pi}{2}\right)}{\cos^{-2}\left(\alpha-\frac{\pi}{2}\right)-1}$
   e. $\frac{\left(1+\tan^2\left(\alpha-\frac{\pi}{2}\right)\right)\left(\sin^{-2}\left(\alpha-\frac{3\pi}{2}\right)-1\right)}{\left(1+\cot^2\left(\alpha+\frac{3\pi}{2}\right)\right)\cos^{-2}\left(\alpha+\frac{\pi}{2}\right)}$
   f. $\frac{\sin^2\left(\alpha+\frac{\pi}{2}\right)-\cos^2\left(\alpha-\frac{\pi}{2}\right)}{\tan^2\left(\alpha+\frac{\pi}{2}\right)-\cot^2\left(\alpha-\frac{\pi}{2}\right)}$
2. Solve the following equations (find all solutions):
   a. $\sin x = \frac{1}{2}$
   b. $\tan x = 1$
   c. $\cos x = \frac{\sqrt{3}}{2}$
   d. $\cos^2 x = \frac{1}{2}$
3. Solve the following equations and inequalities:
   a. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
   b. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
   c. $\sin^2 x - 2\sin x\cos x = 3\cos^2 x$
   d. $\sin 6x + 2 = 2\cos 4x$

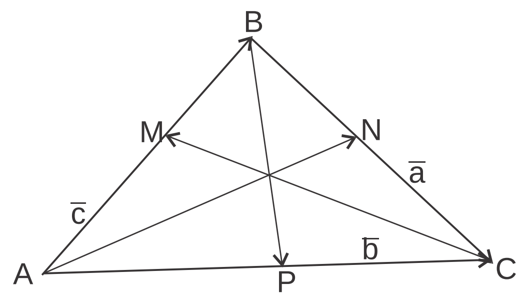
e. $\cot x - \tan x = \sin x + \cos x$
f. $\sin x \geq \pi/2$
g. $\sin x \leq \cos x$

Geometry. Vectors.

Please, complete the previous homework assignments from this year. Review the classwork handout on vectors. Solve the following problems.

Problems.





1. In a triangle ABC, vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$and $\overrightarrow{BC}$ (**c**, **b** and **a**) are the sides. $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$are the medians.

   a. Express vectors $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$ through vectors **c**, **b** and **a.**

   b. Find the sum of vectors $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$**.**

2. Solve the same problem for bisectors $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$ in a triangle ABC.

3. Coxeter, Greitzer, problem #9 to Sec. 2.1 (p. 31): How far away is the horizon as seen from the top of a mountain 1 mile high? (Assume the Earth to be a sphere of diameter 7920 miles.)

4. In a rectangle $ABCD$, A_1, B_1, C_1 and D_1 are the mid-points of sides AB, CD, BC and DA, respectively. M is the crossing point of the segments A_1B_1, and C_1D_1, connecting two pairs of midpoints.

   a. Express vector $\overrightarrow{A_1M}$ through $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$.

   b. Prove that M is the mid-point of segments, A_1B_1 and C_1D_1, i.e. $|A_1M| = |MB_1|$ and $|C_1M| = |MD_1|$.

5. In a parallelogram $ABCD$, find $\overrightarrow{AB} + \overrightarrow{BD} - 2\overrightarrow{AD}$.

6. M is a crossing point of the medians in a triangle ABC. Prove that $\overrightarrow{AM} = \frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.

7. For three points, $A(-1,3)$, $B(2,-5)$ and $C(3,4)$, find the (coordinates of) following vectors,

a. $\overrightarrow{AB} - \overrightarrow{BC}$

b. $\overrightarrow{AB} + \overrightarrow{CB} + \overrightarrow{AC}$

c. $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{CA}$

8. For two triangles, ABC and $A_1B_1C_1$, $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 0$. Prove that medians of these two triangles cross at the same point M.